MATH 4443 **Homework 10** **Due 4/17/18 (Tuesday)**

Problem 1. Two metrics ρ and σ on a set X are said to be *topologically equivalent* if for each $x \in X$ and each number $r > 0$, there is a number $s > 0$ (which in general depends on x and r) such that

$$B_s^\rho(x) \subset B_r^\sigma(x) \quad \text{and} \quad B_s^\sigma(x) \subset B_r^\rho(x) ,$$

where $B_r^\rho(x) := \{y \in X : \rho(x,y) < y\}$ is the open ball of radius r centered at x (and similarly for $B_s^\sigma(x)$, etc.).

(a) Recall that an open set A in a metric space (X,ρ) is defined as a set with the property that, if $x \in A$, then there exists an open ball $B_r^\rho(x)$ that is entirely contained in A.

Prove that topologically equivalent metrics have the same open sets (which can be restated by saying that topologically equivalent metrics induce the same topology), i.e., that every open set $A \subseteq X$ in (X,ρ) is an open set in (X,σ), and that every open set $A \subseteq X$ in (X,σ) is an open set in (X,ρ).

(b) Prove that topologically equivalent metrics have the same closed sets.

Hint: This part of the problem is almost trivial – use the fact that the metric spaces are also topological spaces (with topology induced by the metric), and recall the definition of a closed set in a topological space.

(c) In the rest of this problem you will consider $\mathbb{R}$ endowed with the two different metrics:

$$\rho(x,y) = |x - y| , \qquad \sigma(x,y) = |\mathrm{e}^x - \mathrm{e}^y| .$$

In this part and parts (d)-(g) below you will show that the metrics ρ and σ are topologically equivalent.

Apply the Mean Value Theorem to show that $\sigma(x,y) \leq \alpha\rho(x,y)$ for all $x, y \in \mathbb{R}$ for some constant $\alpha \in \mathbb{R}$. Give an explicit expression for α in terms of x and y.

(d) Use your result from part (c) to show that for every $r > 0$ there exists $s_1 > 0$ such that $B_{s_1}^\rho(x) \subset B_r^\sigma(x)$. In other words, find an expression for s_1 in terms of r and the constant α from part (c) such that $y \in B_{s_1}^\rho(x)$ implies that $y \in B_r^\sigma(x)$.

(e) Show that $\rho(x,y) \leq \beta\sigma(x,y)$ for all $x, y \in \mathbb{R}$ for some constant $\beta \in \mathbb{R}$. Give an explicit expression for β in terms of x and y.

Hint: You can use the Mean Value Theorem similarly to the way you used it in part (c), but you have to obtain an inequality going in opposite direction.

(f) Use your result from part (e) to show that for every $r > 0$ there exists $s_2 > 0$ such that $B_{s_2}^\sigma(x) \subset B_r^\rho(x)$. In other words, find an expression for s_2 in terms of r and the constant β from part (e) such that $y \in B_{s_2}^\sigma(x)$ implies that $y \in B_r^\rho(x)$.

(g) Use your results from parts (d) and (f) to prove that the metrics ρ and σ on X are topologically equivalent. Give an explicit expression for s.

(h) The metric space $(\mathbb{R}, \rho)$ is complete because ρ is the "standard" distance in $\mathbb{R}$, so that the Axiom of Completeness holds. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ given by $x_n = -n$. Clearly, it is not Cauchy in the metric ρ because $\rho(x_n, x_m) \geq 1$ for $n \neq m$.

Now consider the same sequence, $(x_n)_{n\in\mathbb{N}} = (-n)_{n\in\mathbb{N}}$, in the metric space $(\mathbb{R}, \sigma)$. Prove that (x_n) is a Cauchy sequence in $(\mathbb{R}, \sigma)$.

(i) Does (x_n) converge in $(\mathbb{R}, \sigma)$? Is the metric space $(\mathbb{R}, \sigma)$ complete? Discuss the meaning of your observation.

Problem 2. Two metrics ρ and σ on a set X are said to be *equivalent* (or *strongly equivalent*) if there exist constants $C_1 > 0$ and $C_2 > 0$ such that $C_1\rho(x, y) \leq \sigma(x, y) \leq C_2\rho(x, y)$ for all $x, y \in X$.

(a) Prove that equivalent metrics are topologically equivalent.

(b) Prove that equivalent metrics have the same Cauchy sequences.

(c) Use Problem 1(h,i) to construct topologically equivalent metrics that are not equivalent.

(d) [**Food for Thought only!**] Think about the meaning of the following statement:

> The continuity of a function $f : X \to Y$ (where (X, ρ) and (Y, τ) are metric spaces) is preserved if either ρ or τ is replaced by a topologically equivalent metric, but uniform continuity is preserved only if either ρ or τ is replaced by an equivalent metric.

Problem 3. Consider $\mathbb{R}^2$ endowed with the Euclidean norm, $\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2}$.

(a) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{x_1^3}{x_1^2 + x_2^2} & \text{if } (x_1, x_2) \neq (0,0) \ , \\ 0 & \text{if } (x_1, x_2) = (0,0) \ . \end{cases}$$

Directly from the ε-δ definition of continuity, prove that f is continuous. The continuity of f in $\mathbb{R}^2\backslash\{(0,0)\}$ is clear (because there f is a rational function with a strictly positive denominator), so you only have to prove its continuity at $(0,0)$.

(b) Show that the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{x_1 x_2}{x_1^2 + x_2^2} & \text{if } (x_1, x_2) \neq (0,0) \ , \\ 0 & \text{if } (x_1, x_2) = (0,0) \end{cases}$$

is not continuous at $(0,0)$.

Problem 4. Let $C([a,b])$ stand for the set of continuous functions $f : [a,b] \to \mathbb{R}$.

(a) Show that

$$\rho(f,g) = \int_a^b |f(x) - g(x)| \, \mathrm{d}x$$

is a metric on $C([a,b])$.

(b) Convince me that the metric space $(C([a,b]), \rho)$ is not complete.

Problem 5. Consider the map

$$\vec{f} : \mathbb{R}^2 \to \mathbb{R}^3 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} x_1 x_2 \\ 5x_1 + x_2^3 \\ \frac{1}{x_2} \end{bmatrix} .$$

Directly from the definition of derivative, compute the derivative $D\vec{f}(\mathbf{x}) \in L(\mathbb{R}^2, \mathbb{R}^3)$.

Food for Thought: The Contraction Mapping Theorem proved in Problem 7 of Homework 9 for the particular case of functions from $\mathbb{R}$ to $\mathbb{R}$ holds for functions from a complete metric space to itself. Consider a complete metric space (X, ρ) and a function $f : X \to X$ satisfying

$$\rho(f(x), f(y)) \leq c\rho(x,y) \qquad \forall x, y \ X \ ,$$

and $c \in [0,1)$ is a constant. Think how you would generalize the statements in all parts of Problem 7 of Homework 9 to this case. Why do we require completeness?