MATH 5163 **Homework 10** **Due Fri, 5/2/14, 5 p.m.**

Problem 1. [Parallelogram identity]

(a) Let H be an inner product linear space. Prove the *parallelogram identity*,

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right) \qquad \text{for any } u, v \in H \ .$$

(b) Consider the functions $f(x) = x$ and $g(x) = 1 - x$ defined on $[0,1]$ to show that $C[0,1]$ (with the sup-norm) is *not* an inner product linear space.

Problem 2. [The space $H^1(\Omega)$]
Let $\Omega = B_1(\mathbf{0})$ be the unit ball in $\mathbb{R}^2$ centered at the origin $\mathbf{0}$, and let $u(\mathbf{x}) = |\mathbf{x}|^\alpha$ ($\mathbf{x} \neq \mathbf{0}$) for some $\alpha \in \mathbb{R}$.

(a) For which values of α does the function u belong to $L^2(\Omega)$?

(b) For which values of α does the function u belong to $H^1(\Omega)$?

Problem 3. [Weak solution of Poisson's equation on $[0,1]$]
Let $a \in (0,1)$ and δ_a be the "δ-function concentrated at a", i.e., the distribution $\delta_a \in \mathscr{D}'([0,1])$ defined by

$$\langle \delta_a, \phi \rangle = \phi(a) \ , \qquad \phi \in \mathscr{D}([0,1]) \ .$$

(a) Explain why $H^{-1}([0,1])$.

(b) Find the weak solution $u \in H^1_0([0,1])$ of the Poisson's equation

$$-\Delta u = \delta_a \ .$$

Remark: The solution of this problem has a simple physical meaning: it describes the equilibrium shape of a string with both ends firmly attached at the points $x = 0$ and $x = 1$, with a unit load attached to the string at the point $x = a$.

Problem 4. [The space $H^{-1}(\Omega)$]
Let $\Omega = B_1(\mathbf{0})$ be the (open) unit ball in $\mathbb{R}^2$ centered at the origin $\mathbf{0}$, $\Lambda \subset \Omega$ be a compact subset of Ω, and χ_Λ be the indicator function of Λ.

(a) How is the distribution $\nabla\chi_\Lambda$ defined? (Just follow the general definition, inspired by Green's formulas.)

(b) In the light of Theorem 7.6 from page 401 of Salsa's book, explain why $\nabla \chi_\Lambda$ is an element of $H^{-1}(\Omega)$.

(c) Let $a \in (0, \frac{1}{2})$ and $\Lambda = \overline{B_a(\mathbf{0})} \subset \Omega$ be the closed ball of radius a. Let $\mathbf{u} \in H_0^1(\Omega, \mathbb{R}^2)$ be a defined by

$$\mathbf{u}(\mathbf{x}) = \begin{cases} (\frac{1}{2} - |\mathbf{x}|^2)^5 \, \mathbf{i} & \text{for } |\mathbf{x}| \le \frac{1}{2} \\ \mathbf{0} & \text{for } |\mathbf{x}| \in (\frac{1}{2}, 1) \end{cases}$$

(where $\mathbf{i}$ is the unit vector in positive x-direction). Compute the value of $\langle \nabla \chi_\Lambda, \mathbf{u} \rangle$.