MATH 5453 Homework 10 Due Thu, Nov 13

Additional problem 1.

(a) Prove that the Riemann integral $\int_0^\infty \frac{\sin(2\pi x)}{x}\,dx$ exists (you need not compute its value, which, incidentally, is $\frac{\pi}{2}$).

Hint: What do you know about the convergence of alternating series whose terms decrease by absolute value?

(b) Prove that the function $f\colon [0,\infty) \to \mathbb{R} : x \mapsto \frac{\sin(2\pi x)}{x}$ is not Lebesgue integrable.

Additional problem 2.

(a) Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Prove that $f_n \to f$ μ-a.e. if and only if, for any $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \mu\left(\left\{x \in X \ : \ \sup_{k\ge n} |f_k(x) - f(x)| \ge \frac{1}{m}\right\}\right) = 0 \ .$$

Please clearly point out where you use the fact that $\mu(X) < \infty$.

Remark: Note that this condition is equivalent to

$$\lim_{n\to\infty} \mu\left(\left\{x \in X \ : \ \sup_{k\ge n} |f_k(x) - f(x)| \ge \epsilon\right\}\right) = 0 \qquad \text{for any } \epsilon > 0 \ ,$$

but is more convenient to use because it is easier to work with countable sets.

Hint: First establish the equalities

$$\begin{aligned}
&\{x \in X : f_n(x) \nrightarrow f(x) \ \text{ as } n \to \infty\} \\
&\qquad = \bigcup_{m\in\mathbb{N}} \left\{x \in X : |f_n(x) - f(x)| \ge \frac{1}{m} \ \text{ for infinitely many } n\right\} \\
&\qquad = \bigcup_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \left\{x \in X : \sup_{k\ge n} |f_k(x) - f(x)| \ge \frac{1}{m}\right\} \ .
\end{aligned}$$

(b) Use your result from part (a) to show that, if $\mu(X) < \infty$, the μ-almost everywhere convergence implies convergence in measure.

(c) Give an example of a measure space $(X, \mathcal{M}, \mu)$ and a sequence of functions $\{f_n\}$ on X that converges almost everywhere but not in measure (clearly, the measure of X must be infinite).

Additional problem 3.

Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of functions $g_n : X \to \mathbb{R}$, and $g : X \to \mathbb{R}$. How are the sets

$$\{x \in X : g_j(x) \nrightarrow g(x) \;\text{ as } j \to \infty\}\ ,$$

$$\bigcup_{m\in\mathbb{N}} \left\{x \in X : |g_j(x) - g(x)| \geq \frac{1}{m} \;\text{ for infinitely many } j\right\}\ ,$$

$$\bigcup_{m\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} \bigcup_{\ell\geq j} \left\{x \in X : |g_\ell(x) - g(x)| \geq \frac{1}{m}\right\}\ ,$$

and

$$\bigcup_{m\in\mathbb{N}} \limsup_{j\to\infty} \left\{x \in X : |g_j(x) - g(x)| \geq \frac{1}{m}\right\}$$

related? Support your answer with brief explanations.

(b) Prove that a necessary and sufficient condition for the function sequence $\{f_n\}_{n=1}^{\infty}$ to converge to f in measure is that each subsequence $\{f_{n_k}\}_{k=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ contains a subsequence $\{f_{n_{k_j}}\}_{j=1}^{\infty} \subset \{f_{n_k}\}_{k=1}^{\infty}$ such that $f_{n_{k_j}} \to f$ a.e. as $j \to \infty$.

Hint: The necessity of the condition follows easily from Theorem 2.30.

To prove that the condition is sufficient, choose a subsequence $\{n_{k_j}\}_j \subset \{n_k\}_k$ such that $\mu(\{x \in X : |f_{n_k}(x) - f(x)| \geq \frac{1}{2^j}\}) < \frac{1}{2^j}$ for all $k \geq k_j$, use some of the facts proved in part (a), and apply the first Borel-Cantelli Lemma (Additional Problem 2 in Homework 3).

Additional problem 4.

(a) Prove that, if $\mu(X) < \infty$ and $h : X \to [-\infty, \infty]$ is a measurable function that takes finite values on a set of full measure, then

$$\lim_{k\to\infty} \mu\left(\{x \in X : |h(x)| > k\}\right) = 0\ .$$

Hint: Define the sets $E := \{x \in X : |h(x)| = \infty\}$, and $E_k := \{x \in X : |h(x)| > k\}$ for $k \in \mathbb{N}$, and apply some fundamental properties of measures.

(b) Give an example of a space X with $\mu(X) = \infty$ and a measurable function $f : X \to \mathbb{R}$ for which $\mu\left(\{x \in X : |h(x)| > k\}\right)$ does not go to zero as $k \to \infty$.

(c) In Additional Problem 1 of Homework 9 you showed that, if $f_n \to f$ in measure and $g_n \to g$ in measure, and $\alpha \in \mathbb{R}$ is a constant, then then $\alpha f_n \to \alpha f$ in measure, and

$f_n + f_n \to f + g$ in measure. The product of two function sequences, however, does not converge in measure unless some additional condition is imposed.

Prove that, if $\mu(X) < \infty$, then $f_n g_n \to fg$ in measure.

Hint: Start with proving the inequality

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x) - f(x)|\,|g_n(x) - g(x)| \\ &\quad + |f(x)|\,|g_n(x) - g(x)| + |g(x)|\,|f_n(x) - f(x)|\ . \end{aligned}$$

Use this inequality to give an upper bound on $\mu\left(\{x : |f_n(x)g_n(x) - f(x)g(x)| \geq \epsilon\}\right)$ in terms of the measures of the set $\left\{x : |f_n(x)g_n(x) - f(x)g(x)| \geq \frac{\epsilon}{3}\right\}$, and the sets $\left\{x : |f(x)|\,|g_n(x) - g(x)| \geq \frac{\epsilon}{3}\right\}$ and $\left\{x : |g(x)|\,|f_n(x) - f(x)| \geq \frac{\epsilon}{3}\right\}$. Use the result of part (a) to bound the measures of the last two sets.

(d) Consider the functions $f_n : [0,\infty) \to \mathbb{R} : x \mapsto f_n(x) = x + \frac{1}{n}$, and $f : [0,\infty) \to \mathbb{R} : x \mapsto f(x) = x$. Demonstrate that $f_n \to f$ in measure, but $f_n^2 \nrightarrow f^2$ in measure. Explain why this does not contradict the fact proved in part (c).

Additional problem 5.

Let $\mathcal{E}$ be a collection of subsets of some set X satisfying the following properties:

(i) $\emptyset \in \mathcal{E}$;

(ii) if $E \in \mathcal{E}$ and $F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$;

(iii) the complement of each set $E \in \mathcal{E}$ is a finite disjoint union of sets from $\mathcal{E}$.

Let $\mathcal{A}$ be the collection of finite disjoint unions of members of $\mathcal{E}$. Use induction to prove that $\mathcal{A}$ is an algebra.

Remark: We used this fact in the lectures to show that the collection $\mathcal{A}$ of all finite disjoint unions of measurable rectangles $A \times B$ (where $A \in \mathcal{M}$, $B \in \mathcal{N}$) is an algebra; the σ-algebra generated by $\mathcal{A}$ is by definition the product σ-algebra $\mathcal{M} \otimes \mathcal{N}$.