MATH 5453 Homework 10a <u>Not</u> Due Thu, Nov 27

Problems 46, 48 from Section 2.5 of the book.

Problems 2, 3, 4, 5, 7 from Section 3.1 of the book.

Additional problem 0. Think about the proofs of all small remarks made in class (every time you heard the word "obviously" or "clearly") about the concept of a signed measure, its positive, negative, and total variations, mutually singular signed measures, and a signed measure mutually singular with respect of a positive measure.

Additional problem 1. Let $E = [1, \infty) \times (0, 1] \subset \mathbb{R} \times \mathbb{R}$, and m stand for the Lebesgue measure on $\mathbb{R}$. Consider the function

$$f : E \to \mathbb{R} : (x, y) \mapsto f(x, y) = \mathrm{e}^{-xy} - 2\mathrm{e}^{-2xy} .$$

(a) <u>*Optional!*</u> Compute the integral

$$\int_{[1,\infty)} \left(\int_{(0,1]} f(x, y)\, \mathrm{d}m(y) \right) \mathrm{d}m(x)$$

treating it as a Riemann integral. You can express the answer in terms of the function $\Phi(z) := \int_1^z \frac{1}{t}\, \mathrm{e}^{-zt}\, \mathrm{d}t$.

(b) <u>*Optional!*</u> Compute the integral

$$\int_{(0,1]} \left(\int_{[1,\infty)} f(x, y)\, \mathrm{d}m(x) \right) \mathrm{d}m(y)$$

treating it as a Riemann integral. Again, express the answer in terms of the function Φ, and use the fact that $\int_0^\infty \frac{\mathrm{e}^{-at} - \mathrm{e}^{-bt}}{t}\, \mathrm{d}t = \ln \frac{b}{a}$ $(a > 0,\ b > 0)$.

(c) Explain why your answers in parts (a) and (b) were different, i.e., why the Fubini-Tonelli Theorem did not work.

Additional problem 2. Recall the following strategy of using Fubini-Tonelli Theorem together in order to reverse the order of integration in a double integral. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, and $f : X \times Y \to \mathbb{C}$ be an $\mathcal{M} \otimes \mathcal{N}$-measurable function. In order to have

$$\int f\, \mathrm{d}(\mu \times \nu) = \int \int f\, \mathrm{d}\mu\, \mathrm{d}\nu = \int \int f\, \mathrm{d}\nu\, \mathrm{d}\mu , \tag{1}$$

it will be enough (because of Fubini) to show that $f \in L^1(\mu \times \nu)$, i.e., that $\int |f|\, \mathrm{d}(\mu \times \nu) < \infty$. Since f is $\mathcal{M} \otimes \mathcal{N}$-measurable, $|f|$ is also $\mathcal{M} \otimes \mathcal{N}$-measurable, and also $|f| \geq 0$, so that $|f| \in L^+(\mu \times \nu)$. Thanks to Tonelli, we have

$$\int |f|\, \mathrm{d}(\mu \times \nu) = \int \int |f|\, \mathrm{d}\mu\, \mathrm{d}\nu = \int \int |f|\, \mathrm{d}\nu\, \mathrm{d}\mu ,$$

so that if we show that either $\int\int |f| \,\mathrm{d}\mu \,\mathrm{d}\nu$ or $\int\int |f| \,\mathrm{d}\nu \,\mathrm{d}\mu$ is finite, this will imply that $f \in L^1(\mu \times \nu)$, and, hence, (1) will hold.

Consider the function

$$f : [0, \infty) \times [0, \infty) \to \mathbb{R} : (x, y) \mapsto f(x, y) = \frac{\mathrm{e}^{-x(y+1)} \sin(xy)}{\sqrt{y}} \ .$$

Apply the above strategy to show that for this function and the Lebesgue measure on $\mathbb{R}^2$ the equation (1) holds.

Remark: The order in which you try to integrate in (1) matters, so do the things in the easiest possible way.