MATH 3413 Homework 11 Due Tue, Nov 21

Sec. 9.5: problems 1, 2, 7, 10, 11.

Hints: In problem 7 the easiest way to find the Fourier expansion of the corresponding function is to use some trigonometric identity.

In problems 10 and 11, you will have to use two different expansions of the function $f(x) = 4x$, $x \in (0, 10)$: in one case you have to extend it as an *odd* function on the interval $(-10, 10)$, and in the other case you have to extend it as an *even* function on the interval $(-10, 10)$; *you* have to decide which expansion to use in which case. Use the fact that the expansion of the function $h(x) = x$ for $x \in (0, L)$, extended to $(-L, L)$ as an *even* function is

$$x = \frac{L}{2} - \frac{4L}{\pi^2}\left(\frac{1}{1^2}\cos\frac{\pi x}{L} + \frac{1}{3^2}\cos\frac{3\pi x}{L} + \frac{1}{5^2}\cos\frac{5\pi x}{L} + \frac{1}{7^2}\cos\frac{7\pi x}{L} + \cdots\right) ;$$

the expansion of $h(x) = x$ for $x \in (0, L)$, extended to $(-L, L)$ as an *odd* function is

$$x = \frac{2L}{\pi}\left(\frac{1}{1}\sin\frac{\pi x}{L} - \frac{1}{2}\sin\frac{2\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} - \frac{1}{4}\sin\frac{4\pi x}{L} + \frac{1}{5}\sin\frac{5\pi x}{L} - \cdots\right)$$

(these expansions are obtained in Example 1 in Sec. 9.3).

Additional problem 1. Solve the following initial-boundary value problem for the heat equation

$$\begin{aligned}
&u_t = ku_{xx} , \qquad x \in [0, L] , \ t \in [0, \infty) \\
&u(0, t) = A , \qquad u(L, t) = B \qquad \text{for } t \in [0, \infty) \\
&u(x, 0) = f(x) \qquad \text{for } x \in [0, L] ,
\end{aligned}$$

where A and B are constants, in general nonzero. Since we know how to solve problems with two homogeneous (i.e., zero) boundary conditions on a pair of opposite sides, we would be able to solve this problem if $A = B = 0$. One method to reduce the given problem to a problem of the type we like is to set

$$u(x, t) = v(x, t) + \ell(x) ,$$

where $\ell(x) := \alpha x + \beta$ is a linear function that – for an appropriate choice of the constants α and β – can be made to satisfy the boundary conditions $\ell(0) = A$, $\ell(L) = B$. Choose the constants α and β appropriately, *show* that the initial-boundary value problem for the function $v(x, t)$ is

$$\begin{aligned}
&v_t = kv_{xx} , \qquad x \in [0, L] , \ t \in [0, \infty) \\
&v(0, t) = 0 , \qquad v(L, t) = 0 \qquad \text{for } t \in [0, \infty) \\
&v(x, 0) = f(x) - \ell(x) \qquad \text{for } x \in [0, L] ,
\end{aligned}$$

and solve this problem. Assume that the sine-Fourier expansion of the function $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} f_n \sin \frac{n\pi x}{L} \ .$$

Be careful in formulating the problem for $v(x,t)$; in particular, the initial condition for $v(x,t)$ will differ from the one for $u(x,t)$. Here are two facts that you will need (when you expand the function $\ell(x)$)

$$1 = \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \sin \frac{n\pi x}{L} \ , \qquad x = \frac{2L}{\pi} \sum_{n=1,3,5,\ldots} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{L} \ .$$

Show that as $t \to \infty$, the function $v(x,t)$ will tend to 0, so that the asymptotic behavior of the temperature $u(x,t)$ is determined completely by the auxiliary function $\ell(x)$ (recall that the constant k in the heat equation is strictly positive!). Sketch $\lim_{t\to\infty} u(x,t)$ as a function of x.

Additional problem 2. In this problem you will make some predictions about the asymptotic behavior (i.e., when $t \to \infty$) of the solution $u(x,t)$ of the boundary value problem

$$\begin{aligned} &u_t = k\, u_{xx} + \phi(x) \ , \qquad x \in [0,L] \ , \quad t \in [0,\infty) \\ &u(0,t) = 0 \ , \qquad u(L,t) = 0 \qquad \text{for } t \in [0,\infty) \\ &u(x,0) = f(x) \qquad \text{for } x \in [0,L] \ . \end{aligned}$$

Physically, this problem describes the temperature distribution in a rod of length L with insulated side walls and ends at $x = 0$ and $x = L$ kept at zero temperature. The initial temperature in the rod is given by the function $f(x)$ and, more interestingly, there are sources of heat in the rod whose power is given by the function $\phi(x)$ in the PDE.

One can solve this problem completely (which you will do in Additional problem 3 below), but before doing this, try to obtain some information about the behavior of the solution $u(x,t)$ at large times. Since the temperatures at the ends of the rod do not depend on time, and the intensity of the sources of heat is time-independent as well, it is clear that after some initial period of more or less rapid changes, the solution $u(x,t)$ will tend to some time-independent function. Let us call this function $u_\infty(x)$:

$$u_\infty(x) := \lim_{t\to\infty} u(x,t) \ .$$

Since this function does not depend on t, it will be a solution of some *ordinary* differential equation!

(a) From the PDE given in this problem, obtain an ODE for the function $u_\infty(x)$.

(b) From the BCs for $u(x,t)$, obtain BCs for $u_\infty(x)$. Note that the initial condition for $u(x,t)$ will not matter in the limit $t \to \infty$.

(c) Solve the boundary value problem for the asymptotic temperature distribution $u_\infty(t)$ in the case $\phi(x) = -2\sin\dfrac{5\pi x}{L}$.

(d) Sketch the function $u_\infty(x)$. Find the highest and the lowest temperatures in the beam after very long time.

Additional problem 3.

The method of separation of variables can also be used to solve boundary value problems for *non-homogeneous* partial differential equations, like the heat equation with a source of heat of "intensity" $\phi(x,t)$:

$$u_t = ku_{xx} + \phi(x,t) \ .$$

In this problem we will consider one such situation; below I first illustrate the idea on a problem that you already know how to solve.

The solution of the boundary value problem

$$u_t = k\, u_{xx} \ , \qquad x \in [0,L] \ , \quad t \in [0,\infty) \tag{1}$$

$$u(0,t) = 0 \ , \qquad u(L,t) = 0 \qquad \text{for } t \in [0,\infty) \tag{2}$$

$$u(x,0) = f(x) \qquad \text{for } x \in [0,L] \tag{3}$$

can be written as an infinite series of the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\frac{n\pi x}{L} \ . \tag{4}$$

Here the functions $X_n(x) = \sin\frac{n\pi x}{L}$ satisfy an ordinary differential equation coming from the separation of variables, as well as the boundary conditions $X(0) = 0$ and $X(L) = 0$ (which come from the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$ for all $t \in [0,\infty)$).

Now pretend that you do not know how to find the functions $T_n(t)$. One method of finding them is to plug the above series for $u(x,t)$ in the PDE and in the initial condition and from this to derive the ODE for $T_n(t)$ and the corresponding initial condition.

Indeed, from (4) we obtain by differentiating the infinite sum term by term

$$u_t = \sum_{n=1}^{\infty} T_n'(t) \sin\frac{n\pi x}{L} \ ,$$

$$u_x = \sum_{n=1}^{\infty} \frac{n\pi}{L} T_n(t) \cos\frac{n\pi x}{L} \ , \qquad u_{xx} = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 T_n(t) \sin\frac{n\pi x}{L} \ .$$

Plugging these expressions into the PDE $u_t = k\, u_{xx}$, we obtain

$$\sum_{n=1}^{\infty} T_n'(t) \sin\frac{n\pi x}{L} = -k \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 T_n(t) \sin\frac{n\pi x}{L} \ . \tag{5}$$

Similarly, from the initial condition $u(x,0) = f(x)$ we obtain

$$\sum_{n=1}^{\infty} T_n(0) \sin \frac{n\pi x}{L} = \sum_{n=1}^{\infty} f_n \sin \frac{n\pi x}{L} \ , \tag{6}$$

where f_n are the Fourier coefficients of the function $f(x)$ (extended as an *odd* function from $(0, L)$ to $(-L, L)$):

$$f(x) = \sum_{n=1}^{\infty} f_n \sin \frac{n\pi x}{L} \ .$$

From (5) and (6) we obtain that the function $T_n(t)$ must satisfy

$$\begin{aligned} &T_n'(t) + \left(\frac{n\pi}{L}\right)^2 k\, T_n(t) = 0 \\ &T_n(0) = f_n \ . \end{aligned}$$

From here we easily obtain

$$T_n(t) = f_n\, e^{-\left(\frac{n\pi}{L}\right)^2 kt} \ .$$

Now you will have to follow the same pattern to solve the boundary value problem

$$u_t = k\, u_{xx} + \phi(x) \ , \qquad x \in [0, L] \ , \quad t \in [0, \infty) \tag{7}$$

$$u(0,t) = 0 \ , \qquad u(L,t) = 0 \qquad \text{for } t \in [0, \infty) \tag{8}$$

$$u(x,0) = f(x) \qquad \text{for } x \in [0, L] \ . \tag{9}$$

(a) Since the boundary conditions (8) are the same as (2), you should look for a solution of the new boundary value problem (7), (8), (9) in the same form as above (namely, in the form of the series (4)), and you will try to find the functions $T_n(t)$ (of course, these functions will be different from the functions we have found when we solved the problem (1), (2), (3)). Plug this form of $u(x,t)$ into the PDE (7) and write down the ODE that the functions $T_n(t)$ must satisfy. Assume that you know the coefficients ϕ_n of $\phi(x)$ in the expansion

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \sin \frac{n\pi x}{L} \ .$$

(b) Plug the series (4) into the initial condition and write down the initial conditions that must be satisfied by the functions $T_n(t)$.

(c) Solve the initial value problems for the functions $T_n(t)$, and write the solution $u(x,t)$ in the particular case

$$f(x) = 3 \sin \frac{7\pi x}{L} \ , \qquad \phi(x) = -2 \sin \frac{5\pi x}{L} \ .$$

(d) Show that the solution for $u(x,t)$ obtained in (c) tends to the function $u_\infty(x)$ obtained in Additional problem 2(c).