

Additional problem 1:

(a) $\Delta u = 0, \quad x \in [0, a], \quad y \in [0, b]$
 $u(0, y) = 0, \quad u(a, y) = 0$
 $u(x, 0) = 0, \quad u(x, b) = 5 \sin \frac{7\pi x}{a}$.

The solution has the form

$$u(x, y) = \sum_{n=1}^{\infty} Y_n(y) \sin \frac{n\pi x}{a}, \quad (*)$$

where the functions $Y_n(x)$ satisfy

$$Y_n'' - \left(\frac{n\pi}{a}\right)^2 Y_n = 0. \quad (\#)$$

Then the function $u(x, y)$ defined by $(*)$ is a solution of the ~~PDE~~

$$\Delta u = 0$$

and satisfies the BCs

$$u(0, y) = 0, \quad u(a, y) = 0.$$

The general solution of the equation $(\#)$ for $Y_n(y)$ is

$$Y_n(y) = A_n \cosh \frac{n\pi y}{a} + B_n \sinh \frac{n\pi y}{a}.$$

The constants A_n and B_n come from the remaining two ICs; using that

$\cosh 0 = 1, \quad \sinh 0 = 0$, we obtain

$$u(x, 0) = \sum_{n=1}^{\infty} (A_n \cdot 1 + B_n \cdot 0) \sin \frac{n\pi x}{a} = 0$$

$$u(x, b) = \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi b}{a} + B_n \sinh \frac{n\pi b}{a} \right) \sin \frac{n\pi x}{a} \\ = 5 \sin \frac{7\pi x}{a}.$$

From the first condition we obtain that all A_n 's are 0, and the second condition then becomes

$$\sum_{n=1}^{\infty} B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a} = 5 \sin \frac{7\pi x}{a}$$

$$\Rightarrow B_7 = \frac{5}{\sinh \frac{7\pi b}{a}}, \quad B_n = 0 \text{ for } n \neq 7.$$

Putting all this together:

$$u(x, y) = \frac{5}{\sinh \frac{7\pi b}{a}} \sinh \frac{7\pi y}{a} \sin \frac{7\pi x}{a}.$$

(b) Using the hint in the statement of the problem, we impose the condition $Y_n(b) = 0$

(coming from $u(x, b) = 0$) on $Y_n(y)$:

$$Y_n(b) = C_n \cosh \frac{n\pi b}{a} + D_n \sinh \frac{n\pi b}{a} = 0,$$

which implies that

$$C_n = -D_n \frac{\sinh \frac{n\pi b}{a}}{\cosh \frac{n\pi b}{a}} = -D_n \tanh \frac{n\pi b}{a}$$

$$\begin{aligned} \Rightarrow Y_n(b) &= \underbrace{-D_n \tanh \frac{n\pi b}{a}}_{C_n} \cosh \frac{n\pi b}{a} + D_n \sinh \frac{n\pi b}{a} \\ &= -\frac{D_n \tanh \frac{n\pi b}{a}}{\cosh \frac{n\pi b}{a}} \left(\sinh \frac{n\pi b}{a} \cosh \frac{n\pi b}{a} \right. \\ &\quad \left. - \cosh \frac{n\pi b}{a} \sinh \frac{n\pi b}{a} \right) \\ &= E_n \sinh \frac{n\pi(b-y)}{a}, \end{aligned}$$

where we have set $E_n := -\frac{D_n \tanh \frac{n\pi b}{a}}{\cosh \frac{n\pi b}{a}}$ to be the relabelled arbitrary constant. Now we have

$$u(x,y) = \sum_{n=1}^{\infty} E_n \sinh \frac{n\pi(b-y)}{a} \sin \frac{n\pi x}{a},$$

and imposing the remaining BCs yields

$$u(x,0) = \sum_{n=1}^{\infty} E_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a} = \sin \frac{3\pi x}{a}$$

\Rightarrow all E_n except E_3 are 0, and

$$E_3 = \frac{1}{\sinh \frac{3\pi b}{a}}.$$

This yields

$$u(x,y) = \frac{1}{\sinh \frac{3\pi b}{a}} \sinh \frac{3\pi(b-y)}{a} \sin \frac{3\pi x}{a}.$$

(c) We have

$$\begin{aligned} \begin{array}{|c|} \hline \sin \frac{7\pi x}{a} \\ \hline \Delta u = 0 \\ \hline \sin \frac{3\pi x}{a} \\ \hline \end{array} &= \begin{array}{|c|} \hline 5 \sin \frac{7\pi x}{a} \\ \hline \Delta u = 0 \\ \hline 0 \\ \hline \end{array} + \begin{array}{|c|} \hline 0 \\ \hline \Delta u = 0 \\ \hline \sin \frac{3\pi x}{a} \\ \hline \end{array} \\ \text{(part (c))} &= \text{(part (a))} + \text{(part (b))}, \end{aligned}$$

so the solution of the problem in part (c) is

$$u(x,y) = \frac{5}{\sinh \frac{7\pi b}{a}} \sinh \frac{7\pi y}{a} \sin \frac{7\pi x}{a} + \frac{1}{\sinh \frac{3\pi b}{a}} \sinh \frac{3\pi(b-y)}{a} \sin \frac{3\pi x}{a}.$$