MATH 3413 Homework 11 Due Fri, Dec 10, 2010

Problem 1. Consider the following problem for the wave equation with air resistance term, with homogeneous Dirichlet BCs on the spatial interval $x \in [0, \pi]$:

$$
\begin{aligned}
&u_{xx} - 10u_t - u_{tt} = 0 \;, \qquad x \in [0,\pi] \;, \quad t \geq 0 \;, \\
&u(0,t) = 0 \;, \quad u(\pi,t) = 0 \;, \qquad t \geq 0 \;, \\
&u(x,0) = -8\sin 3x + 12\sin 13x \;, \quad u_t(x,0) = 0 \;, \qquad x \in [0,\pi] \;.
\end{aligned}
$$

Physically, this problem corresponds to a spring vibrating in air with resistance proportional to the velocity (i.e., to the time derivative $u_t(x,t)$). The coefficient multiplying $u_t(x,t)$ is proportional to the air resistance coefficient.

Because of the homogeneous Dirichlet BCs, it is clear that we should look for an expansion of the unknown function $u(x,t)$ of the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} T_n(t)\sin nx \tag{1}$$

(here $L = \pi$ is the length of the string).

(a) Plug the expansion (1) in the PDE to show that the unknown functions $T_n(t)$ must satisfy the ODEs

$$T_n''(t) + 10\,T_n'(t) + n^2 T_n(t) = 0 \;. \tag{2}$$

(b) The initial conditions for the functions $T_n(t)$ come from the initial conditions for $u(x,t)$. Plug the expansion (1) into the initial conditions for $u(x,t)$ to show that $T_n(0)$ and $T_n'(0)$ are zero for all n except for $n = 3$ and $n = 13$. What are the initial conditions $T_3(0)$ and $T_3'(0)$ for $T_3(t)$, and the initial conditions $T_{13}(0)$ and $T_{13}'(0)$ for $T_{13}(t)$?

(c) Since the ODEs (2) are homogeneous (i.e., have zero right-hand sides), the solutions for all functions $T_n(t)$ with n not equal to 3 or 13 will be identically equal to zero.

Solve the IVP for the function $T_3(t)$.

(d) Solve the IVP for the function $T_{13}(t)$.

(e) Write down the solution,

$$u(x,t) = T_3(t)\sin 3x + T_{13}(t)\sin 13x \;,$$

with the functions $T_3(t)$ and $T_{13}(t)$ found in parts (c) and (d).

(f) From the physical interpretation of the problem, what would you expect the asymptotic position of the string to be. No calculation is needed here, only a couple of sentences of explanation.

(g) Does the solution found in part (e) behave as you predicted on physical grounds in part (f)?

Problem 2. In this problem you will make some predictions about the asymptotic behavior (i.e., when $t \to \infty$) of the solution $u(x,t)$ of the boundary value problem

$$\begin{aligned} &u_t = \alpha^2\, u_{xx} + \phi(x) \ , \qquad x \in [0,L] \ , \quad t \in [0,\infty) \\ &u(0,t) = 0 \ , \qquad u(L,t) = 0 \qquad \text{for } t \in [0,\infty) \\ &u(x,0) = f(x) \qquad \text{for } x \in [0,L] \ . \end{aligned}$$

Physically, this problem describes the temperature distribution in a rod of length L with insulated side walls and ends at $x = 0$ and $x = L$ kept at zero temperature. The initial temperature in the rod is given by the function $f(x)$ and, more interestingly, there are sources of heat in the rod whose power is given by the function $\phi(x)$ in the PDE.

One can solve this problem completely (which you will do in Problem 3 below), but before doing this, try to obtain some information about the behavior of the solution $u(x,t)$ at large times. Since the temperatures at the ends of the rod do not depend on time, and the intensity of the sources of heat is time-independent as well, it is clear that after some initial period of more or less rapid changes, the solution $u(x,t)$ will tend to some time-independent function. Let us call this function $u_\infty(x)$:

$$u_\infty(x) := \lim_{t\to\infty} u(x,t) \ .$$

Since this function does not depend on t, it will be a solution of some *ordinary* differential equation!

(a) From the PDE given in this problem, obtain an ODE for the function $u_\infty(x)$.

(b) From the BCs for $u(x,t)$, obtain BCs for $u_\infty(x)$. Note that the initial condition for $u(x,t)$ will not matter in the limit $t \to \infty$.

(c) Solve the boundary value problem for the asymptotic temperature distribution $u_\infty(t)$ in the case $\alpha = 1$, $L = \pi$, $\phi(x) = 2\sin 5x$, $f(x) = \sin 3x$.

(d) Sketch the function $u_\infty(x)$. Find the highest and the lowest temperatures in the rod after very long time.

Problem 3. Now you will find the exact solution of the boundary value problem

$$\begin{aligned} &u_t = \alpha^2\, u_{xx} + \phi(x) \ , \qquad x \in [0,L] \ , \quad t \in [0,\infty) \\ &u(0,t) = 0 \ , \qquad u(L,t) = 0 \qquad \text{for } t \in [0,\infty) \\ &u(x,0) = f(x) \qquad \text{for } x \in [0,L] \ . \end{aligned}$$

This is the same as in Problem 2, but there you only found the asymptotic behavior of $u(x,t)$ as $t \to \infty$, while here you will solve the problem completely.

(a) Because of the boundary conditions, look for a solution of the problem of the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{L} .$$

Assume that the function $\phi(x)$ in the right-hand side of the PDE can be expanded in a sine Fourier series as

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \sin \frac{n\pi x}{L} ,$$

where the coefficients ϕ_n are given by the standard formula, $\phi_n = \frac{2}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L}\, dx$.

Plug these expansions in the partial differential equation to show that the functions $T_n(t)$ satisfy the non-homogeneous ODEs

$$T_n'(t) + \left(\frac{\alpha n\pi x}{L}\right)^2 T_n(t) = \phi_n .$$

(b) Assume that the sine Fourier series of the he initial condition $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} f_n \sin \frac{n\pi x}{L} .$$

Plug the expansion of $u(x,t)$ into the initial condition to show that the initial conditions for the functions $T_n(t)$ are $T_n(0) = f_n$.

(c) Solve the initial value problems for the functions $T_n(t)$ derived in parts (a) and (b).

(d) Using your results from parts (a) and (c), write down the solution $u(x,t)$ of the original boundary value problem.

(e) Write down the solution $u(x,t)$ of the original boundary value problem in the case $\alpha = 1$, $L = \pi$, $\phi(x) = 2\sin 5x$, $f(x) = \sin 3x$ (the same as in Problem 2c above).

(f) Check if the asymptotic (i.e., as $t \to \infty$) behavior of the solution $u(x,t)$ obtained in part (e) behaves as the function $u_\infty(x)$ obtained in Problem 2d.

Additional problem 1. (***Not to be turned in; the solution is on the web-site*!**) Consider the problem about the stationary temperature distribution in the rectangle $(x,y) \in [0,a] \times [0,b]$ if there are no sources of heat in the rectangle (hence the temperature $u(x,y)$ satisfies Laplace's equation $\Delta u = 0$), and the temperature at the sides of the rectangle is maintained as follows:

$$u(0,y) = 0 , \qquad u(a,y) = 0 \qquad \text{for } y \in [0,b]$$
$$u(x,0) = \sin \frac{3\pi x}{a} , \qquad u(x,b) = 5 \sin \frac{7\pi x}{a} \qquad \text{for } x \in [0,a] .$$

(a) Solve the boundary value problem

$$\begin{aligned} &\Delta u = 0\ , \qquad (x,y) \in [0,a] \times [0,b] \\ &u(0,y) = 0\ , \qquad u(a,y) = 0 \qquad \text{for } y \in [0,b] \\ &u(x,0) = 0\ , \qquad u(x,b) = 5 \sin \frac{7\pi x}{a} \qquad \text{for } x \in [0,a]\ . \end{aligned}$$

(b) Solve the boundary value problem

$$\begin{aligned} &\Delta u = 0\ , \qquad (x,y) \in [0,a] \times [0,b] \\ &u(0,y) = 0\ , \qquad u(a,y) = 0 \qquad \text{for } y \in [0,b] \\ &u(x,0) = \sin \frac{3\pi x}{a}\ , \qquad u(x,b) = 0 \qquad \text{for } x \in [0,a]\ . \end{aligned}$$

Hint: Let $Y_n(y)$ stands for the functions in the expansion

$$u(x,y) = \sum_{n=1}^{\infty} Y_n(y)\, X_n(x)\ ,$$

where because of the homogeneous boundary conditions at $x = 0$ and $x = a$ the functions $X_n(x)$ are given by $X_n(x) = \sin \frac{n\pi x}{a}$. Then the general solution of the ODE for $Y_n(y)$ is

$$Y_n(y) = C_n \cosh \frac{n\pi y}{a} + D_n \sinh \frac{n\pi y}{a}\ .$$

Show that the homogeneous boundary condition at $y = b$ implies that

$$\begin{aligned} Y_n(y) &= E_n \left(\sinh \frac{n\pi b}{a} \cosh \frac{n\pi y}{a} - \cosh \frac{n\pi b}{a} \sinh \frac{n\pi y}{a} \right) \\ &= E_n \sinh \frac{n\pi (b-y)}{a} \end{aligned}$$

(where E_n are constants arbitrary at the moment); here we have used the fact that hyperbolic sine satisfies

$$\sinh(\alpha \pm \beta) = \sinh\alpha \cosh\beta \pm \cosh\alpha \sinh\beta\ .$$

Now impose the remaining boundary condition to find the constants E_n (of which only one will be non-zero).

(c) Since the equation is linear and homogeneous (i.e., with a zero right-hand side), the principle of superposition holds similarly to the case of ordinary differential equations. Using this fact, write down the solution of the boundary value problem

$$\begin{aligned} &\Delta u = 0\ , \qquad (x,y) \in [0,a] \times [0,b] \\ &u(0,y) = 0\ , \qquad u(a,y) = 0 \qquad \text{for } y \in [0,b] \\ &u(x,0) = \sin \frac{3\pi x}{a}\ , \qquad u(x,b) = 5 \sin \frac{7\pi x}{a} \qquad \text{for } x \in [0,a]\ . \end{aligned}$$