MATH 5453 Homework 11 Not Due Thu, Dec 11

Problems 8, 10, 13, 16 from Section 3.2 of the book.

Additional problem 1. Consider the increasing, right-continuous function

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \ , \\ 2 + 3x & \text{if } x \geq 0 \ , \end{cases}$$

and let ν_F be the associated Borel measure on $\mathbb{R}$, as in Section 1.5. Find the Lebesgue decomposition (see Theorem 3.8) of ν_F with respect to:

(a) $\mu = m$, the Lebesgue measure on $\mathbb{R}$;

(b) $\mu = \delta$, the Dirac measure at 0.

In other words, write ν_F as $\nu_F = \rho + \lambda$, where $\rho \ll \mu$, $\lambda \perp \mu$. Identify clearly the measures λ and ρ in each case.

Additional problem 2. We know that, roughly speaking, functions from $L^1(\mathbb{R}, m)$ have to be "small at infinity" (m is the Lebesgue measure on $\mathbb{R}$). Below you will explore this statement in more detail.

(a) Give an example of a function $f \in L^1(\mathbb{R}, m)$ such that $f(x)$ does *not* converge to 0 as $x \to \infty$, and this remains true even if f is changed on a set of Lebesgue measure zero.

(b) Although $f \in L^1(\mathbb{R}, m)$ does not imply that $\lim_{x\to\infty} f(x) = 0$ (even in the sense of part (a)), the following statement holds. If $f \in L^1(\mathbb{R}, m)$, then for any $\epsilon > 0$, we can find a set $E \in \mathcal{L}$ and a number $x_0 \in \mathbb{R}$ such that $m(E) < \epsilon$ and

$$|f(x)| < \epsilon \qquad \text{for all } x \geq x_0 \ , \ x \notin E \ .$$

(c) Prove that part (b) also implies that if $f \in L^1(\mathbb{R}, m)$ and $\epsilon > 0$, we can find a set $F \in \mathcal{L}$ so that $m(F) < \epsilon$ and

$$\chi_{F^c}(x) f(x) \to 0 \qquad \text{as } x \to \infty \ .$$

Additional problem 3. Suppose that $\mathbb{R}$ is a continuous function on $\mathbb{R}$, and $f \in L^1(\mathbb{R}, m)$ (m is the Lebesgue measure on $\mathbb{R}$). Prove that

$$\lim_{n\to\infty} \int_{[0,1]} \left| f\left(x + \frac{1}{n}\right) - f(x) \right| \mathrm{d}x = 0 \ .$$