MATH 5763 Homework 11 Due Fri, 5/6/11

Problem 1. Let $X = \{X_t\}_{t\in\mathbb{R}}$ be a stationary Gaussian process such that the joint distribution of X_t and X_s is binormal with vector of means $\mathbf{m} = (0,0)$ and covariance matrix

$$\mathbf{K} = \begin{pmatrix} \operatorname{Var} X_t & \operatorname{Cov}(X_t, X_s) \\ \operatorname{Cov}(X_t, X_s) & \operatorname{Var} X_s \end{pmatrix} = \begin{pmatrix} \sigma_t^2 & \sigma_t\sigma_s\rho_X(t-s) \\ \sigma_t\sigma_s\rho_X(t-s) & \sigma_s^2 \end{pmatrix} .$$

Throughout this problem, we assume (without loss of generality) that $s \le t$. The autocorrelation function ρ_X depends only on $t-s$ because of the stationarity of the process. Recall that the autocovariance function of the process equals $C_X(t-s) = \operatorname{Cov}(X_t, X_s) = \sigma_t\sigma_s\rho_X(t-s)$. In this problem you have to show that the autocovariance function of the process $X^2 := \{X_t^2\}_{t\in\mathbb{R}}$ is $C_{X^2}(t) = 2C_X(t)^2$.

(a) From the results you obtained in Problem 1 of Homework 10, the identities

$$\mathbb{E}[X_t|X_s] = \rho_X(t-s)\,\frac{\sigma_t}{\sigma_s}\,X_s\ , \qquad \operatorname{Var}(X_t|X_s) = \sigma_t^2\left(1-\rho_X(t-s)^2\right) .$$

follow directly. Explain how you obtain these equalities; please be specific.

(b) From the identities obtained in (a), show that

$$\mathbb{E}[X_t^2|X_s] = \sigma_t^2\left(1-\rho_X(t-s)^2 + \frac{\rho_X(t-s)^2}{\sigma_s^2}\,X_s^2\right) .$$

(c) Use your result from (b) and the fact that the second and the fourth moments of $X_s \sim N(0,\sigma_s^2)$ are $\mathbb{E}[X_s^2] = \sigma_s^2$ and $\mathbb{E}[X_s^4] = 3\sigma_s^4$ (these can be computed by a direct integration, which you do not need to do here), to show that

$$\begin{aligned} \mathbb{E}[X_t^2X_s^2] &= \mathbb{E}[\mathbb{E}[X_t^2X_s^2|X_s]] \\ &= \sigma_t^2\left[\left(1-\rho_X(t-s)^2\right)\sigma_s^2 + \frac{\rho_X(t-s)^2}{\sigma_s^2}\,3\sigma_s^4\right] = \sigma_t^2\sigma_s^2\left(1+2\rho_X(t-s)^2\right) . \end{aligned}$$

What properties of the conditional expectation have you used in the derivation?

Remark: As a consistency check, you can apply the Tower Rule to the identity in (b) to make sure that you will get σ_t^2; you do not need to do this here.

(d) Derive the desired result.

Remark: For the processes $X^k := \{X_t^k\}_{t\in\mathbb{R}}$ the results look more complicated, e.g., it can be shown that $C_{X^3}(t) = 3\left[3+2C_X(t)^2\right]C_X(t)$.

Problem 2. Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{A}, \mathbb{L})$, where $\mathcal{A}$ is the Borel σ-algebra on $[0,1]$, and $\mathbb{L}$ is the Lebesgue measure on $[0,1]$; simply speaking, this means that you know the probability (i.e., measure) of each interval $(a,b) \subseteq [0,1]$, and it is equal to its length, $\mathbb{L}((a,b)) = b - a$ (the probabilities of $(a,b]$, $[a,b)$, and $[a,b]$ are also $b-a$). If $X : [0,1] \to \mathbb{R}$ is a random variable, then $\int_A X(\omega)\,\mathrm{d}\mathbb{L}(\omega) = \int_A X(\omega)\,\mathrm{d}\omega$ (for any $A \in \mathcal{A}$) is the ordinary integral.
Let the random variables X and Y, both on $([0,1], \mathcal{A}, \mathbb{L})$ be defined as follows:

$$X(\omega) = \omega^2 \quad \forall \omega \in [0,1] \ ; \qquad Y(\omega) = \begin{cases} \frac{1}{5} & \text{for } \omega \in [0, \frac{1}{3}] \ , \\ \frac{1}{2} & \text{for } \omega \in (\frac{1}{3}, 1] \ . \end{cases}$$

(a) Find explicitly the σ-algebra $\sigma(Y)$ generated by the random variable Y.

(b) Find $\mathbb{E}[X]$ directly from the definition of expectation, $\mathbb{E}[X] = \int_\Omega X(\omega)\,\mathrm{d}\mathbb{L}(\omega)$.

Remark: Usually the probability measure is not so easy to deal with, so one computes $E[X]$ by changing variables from ω to $x = X(\omega)$, and the formula for the expectation becomes $\mathbb{E}[X] = \int_\mathbb{R} x\,\mathrm{d}F_X(x)$, where $F_X : \mathbb{R} \to [0,1]$ is the distribution (c.d.f.) of X, defined as $F_X(x) = \mathbb{P}(\{X \le x\}) = \mathbb{P}\left(X^{-1}((-\infty, x])\right)$. But in this case $\mathrm{d}\mathbb{L}(\omega) = \mathrm{d}\omega$, so that direct computation of $\mathbb{E}[X]$ is straightforward.

(c) Find the conditional expectation $\mathbb{E}[X|Y]$.

Hint: $\mathbb{E}[X|Y] = \mathbb{E}\big[X\big|[0,\frac{1}{3}]\big]\,\chi_{[0,\frac{1}{3}]} + \mathbb{E}\big[X\big|(\frac{1}{3},1]\big]\,\chi_{(\frac{1}{3},1]} = \frac{1}{27}\,\chi_{[0,\frac{1}{3}]} + \frac{13}{27}\,\chi_{(\frac{1}{3},1]}$; I would like to see your detailed calculations.

(d) Now show me how you compute $\mathbb{E}[\mathbb{E}[X|Y]]$.

Problem 3. This is a continuation of the previous problem. Let Z be a random variable on $([0,1], \mathcal{A}, \mathbb{L})$ defined as

$$Z(\omega) = \begin{cases} \frac{1}{5} & \text{for } \omega \in [0, \frac{1}{3}] \ , \\ \omega & \text{for } \omega \in (\frac{1}{3}, 1] \ . \end{cases}$$

(a) What is the σ-algebra $\sigma(Z)$ generated by the random variable Z? (Since $\sigma(X)$ contains infinitely many sets, just describe them in words.)

(b) Find the conditional expectation $\mathbb{E}[X|Z]$.

(c) Now show me how you compute $\mathbb{E}[\mathbb{E}[X|Z]]$. How should $\mathbb{E}[\mathbb{E}[X|Z]]$ compare with $\mathbb{E}[X]$ and $\mathbb{E}[\mathbb{E}[X|Y]]$?

Problem 4. Recall that if X_t satisfies the stochastic differential equation

$$\mathrm{d}X_t = f(t, X_t)\,\mathrm{d}t + g(t, X_t)\,\mathrm{d}B_t \ , \tag{1}$$

then Itô formula reads

$$\mathrm{d}\Psi(t,X_t) = \left[\frac{\partial \Psi}{\partial t}(t,X_t) + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2}(t,X_t)\, g(t,X_t)^2\right] \mathrm{d}t + \frac{\partial \Psi}{\partial x}(t,X_t)\, \mathrm{d}X_t \ ,$$

or, equivalently (using (1)),

$$\begin{aligned} \mathrm{d}\Psi(t,X_t) \quad &= \quad \left[\frac{\partial \Psi}{\partial t}(t,X_t) + \frac{\partial \Psi}{\partial x}(t,X_t)\, f(t,X_t) + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2}(t,X_t)\, g(t,X_t)^2\right] \mathrm{d}t \\ &\quad + \frac{\partial \Psi}{\partial x}(t,X_t)\, g(t,X_t)\, \mathrm{d}B_t \ . \end{aligned}$$

Here the notations are the following:

$$\frac{\partial \Psi}{\partial x}(t,X_t) := \frac{\partial \Psi}{\partial x}(t,x)\Big|_{x=X_t} \ , \qquad \frac{\partial^2 \Psi}{\partial x^2}(t,X_t) := \frac{\partial^2 \Psi}{\partial x^2}(t,x)\Big|_{x=X_t} \ .$$

(a) Use Itô formula to compute $\mathrm{d}(\mathrm{e}^{at+bB_t})$, where a and b are real constants.

(b) Use your result from (a) to show that the solution of the stochastic differential equation

$$\mathrm{d}X_t = \left(a + \frac{b^2}{2}\right) X_t\, \mathrm{d}t + bX_t\, \mathrm{d}B_t$$

is $X_t = X_0\, \mathrm{e}^{at+bB_t}$.

(c) Use the fact that $\mathbb{E}[\mathrm{e}^{\nu B_t}] = \mathrm{e}^{\frac{\nu^2}{2}t}$ (which we will prove in class) and the solution of the stochastic differential equation obtained in part (b) to show that $\mathbb{E}[X_t] = \mathbb{E}[X_0]\, \mathrm{e}^{a+\frac{b^2}{2}t}$.

(d) Find the variance of X_t.