MATH 3413 Homework 12 Due Tue, Nov 28

Sec. 9.6: problems 2, 5, 6.

Hint: In problem 9.6/6, you will need the Fourier expansion of the function $f(x) = x(\pi - x)$, $x \in (0, \pi)$, extended to a periodic function of period 2π defined for $x \in \mathbb{R}$ either as an even or as an odd function; the corresponding expansions are given below – choose the one that you need:

$$\begin{aligned} x(\pi - x) &= \frac{\pi^2}{6} - \sum_{m=1}^{\infty} \frac{\cos 2mx}{m^2} \ , \\ x(\pi - x) &= \frac{8}{\pi} \sum_{n=1,3,5,\ldots} \frac{\sin nx}{n^3} \ . \end{aligned}$$

Additional problem 1. Consider the problem about the stationary temperature distribution in the rectangle $(x, y) \in [0, a] \times [0, b]$ if there are no sources of heat in the rectangle (hence the temperature $u(x, y)$ satisfies Laplace's equation $\Delta u = 0$), and the temperature at the sides of the rectangle is maintained as follows:

$$\begin{aligned} &u(0, y) = 0 \ , \qquad u(a, y) = 0 \qquad \text{for } y \in [0, b] \\ &u(x, 0) = \sin \frac{3\pi x}{a} \ , \qquad u(x, b) = 5 \sin \frac{7\pi x}{a} \qquad \text{for } x \in [0, a] \ . \end{aligned}$$

(a) Solve the boundary value problem

$$\begin{aligned} &\Delta u = 0 \ , \qquad (x, y) \in [0, a] \times [0, b] \\ &u(0, y) = 0 \ , \qquad u(a, y) = 0 \qquad \text{for } y \in [0, b] \\ &u(x, 0) = 0 \ , \qquad u(x, b) = 5 \sin \frac{7\pi x}{a} \qquad \text{for } x \in [0, a] \ . \end{aligned}$$

(b) Solve the boundary value problem

$$\begin{aligned} &\Delta u = 0 \ , \qquad (x, y) \in [0, a] \times [0, b] \\ &u(0, y) = 0 \ , \qquad u(a, y) = 0 \qquad \text{for } y \in [0, b] \\ &u(x, 0) = \sin \frac{3\pi x}{a} \ , \qquad u(x, b) = 0 \qquad \text{for } x \in [0, a] \ . \end{aligned}$$

Hint: Let $Y_n(y)$ stands for the functions in the expansion

$$u(x, y) = \sum_{n=1}^{\infty} Y_n(y)\, X_n(x) \ ,$$

where because of the homogeneous boundary conditions at $x = 0$ and $x = a$ the functions $X_n(x)$ are given by $X_n(x) = \sin\frac{n\pi x}{a}$. Then the general solution of the ODE for $Y_n(y)$ is

$$Y_n(y) = C_n \cosh\frac{n\pi y}{a} + D_n \sinh\frac{n\pi y}{a} \ .$$

Show that the homogeneous boundary condition at $y = b$ implies that

$$\begin{aligned} Y_n(y) &= E_n \left(\sinh\frac{n\pi b}{a}\cosh\frac{n\pi y}{a} - \cosh\frac{n\pi b}{a}\sinh\frac{n\pi y}{a}\right) \\ &= E_n \sinh\frac{n\pi(b-y)}{a} \end{aligned}$$

(where E_n are constants arbitrary at the moment); here we have used the fact that hyperbolic sine satisfies

$$\sinh(\alpha \pm \beta) = \sinh\alpha\cosh\beta \pm \cosh\alpha\sinh\beta \ .$$

Now impose the remaining boundary condition to find the constants E_n (of which only one will be non-zero).

(c) Since the equation $\Delta u = 0$ is linear and homogeneous (i.e., with a zero right-hand side), the principle of superposition hold similarly to the case of ordinary differential equations. Using this fact, write down the solution of the boundary value problem

$$\begin{aligned} &\Delta u = 0 \ , \qquad (x, y) \in [0, a] \times [0, b] \\ &u(0, y) = 0 \ , \qquad u(a, y) = 0 \qquad \text{for } y \in [0, b] \\ &u(x, 0) = \sin\frac{3\pi x}{a} \ , \qquad u(x, b) = 5\sin\frac{7\pi x}{a} \qquad \text{for } x \in [0, a] \ . \end{aligned}$$

Additional problem 2. You have already solved several boundary value problems for the heat, wave, and Laplace equations by the method of separation of variables, and, no doubt, have noticed how similar their solutions looked. In all parts of the problem below, use your experience to write quickly the solutions of the boundary value problems. Note that, having solved (a) and (b), you will be able to find the solution of (c) *very* easily.

(a) Solve the boundary value problem

$$\begin{aligned} &\Delta u = 0 \ , \qquad x \in [0, a] \ , \quad y \in [0, b] \\ &u(0, y) = 0 \ , \qquad u(a, y) = 5\sin\frac{9\pi y}{b} \qquad \text{for } y \in [0, b] \\ &u(x, 0) = 0 \ , \qquad u(x, b) = 0 \qquad \text{for } x \in [0, a] \ . \end{aligned}$$

(b) Solve the boundary value problem

$$\begin{aligned} &\Delta u = 0 \ , \qquad x \in [0, a] \ , \quad y \in [0, b] \\ &u(0, y) = 0 \ , \qquad u(a, y) = 0 \qquad \text{for } y \in [0, b] \\ &u(x, 0) = -4\sin\frac{3\pi x}{a} \ , \qquad u(x, b) = 0 \qquad \text{for } x \in [0, a] \ . \end{aligned}$$

(c) Solve the boundary value problem

$$\begin{aligned}
&\Delta u = 0 \ , \qquad x \in [0,a] \ , \quad y \in [0,b] \\
&u(0,y) = 0 \ , \qquad u(a,y) = 5\sin\frac{9\pi y}{b} \qquad \text{for } y \in [0,b] \\
&u(x,0) = -4\sin\frac{3\pi x}{a} \ , \qquad u(x,b) = 0 \qquad \text{for } x \in [0,a] \ .
\end{aligned}$$

Additional problem 3. Solve the problems below by the method of separation of variables (as in Additional problem 2, do not derive in detail things that we have derived and used before).

(a) Solve the Dirichlet boundary value problem for the heat equation

$$\begin{aligned}
&u_t = k\, u_{xx} \ , \qquad x \in [0,L] \ , \quad t \in [0,\infty) \\
&u(0,t) = 0 \ , \qquad u(L,t) = 0 \qquad \text{for } t \in [0,\infty) \\
&u(x,0) = -4\sin\frac{3\pi x}{L} \qquad \text{for } x \in [0,L] \ .
\end{aligned}$$

(b) Solve the Neumann boundary value problem for the wave equation

$$\begin{aligned}
&u_{tt} = c^2\, u_{xx} \ , \qquad x \in [0,L] \ , \quad t \in [0,\infty) \\
&u_x(0,t) = 0 \ , \qquad u_x(L,t) = 0 \qquad \text{for } t \in [0,\infty) \\
&u(x,0) = 0 \ , \qquad u_t(x,0) = 2 - 4\cos\frac{3\pi x}{L} \qquad \text{for } x \in [0,L] \ .
\end{aligned}$$

Additional problem 4. Solve the boundary value problem

$$\begin{aligned}
&\Delta u = 0 \\
&u(0,y) = 0 \ , \qquad u(a,y) = 0 \qquad \text{for } y \in [0,b] \\
&u(x,0) = \sin\frac{3\pi x}{a} \qquad \text{for } x \in [0,a] \ .
\end{aligned}$$

in the semi-infinite strip $(x,y) \in [0,a] \times [0,\infty)$. From physical point of view it is quite clear that we have to also impose the condition $\lim_{y\to\infty} u(x,y) = 0$.

Hint: When you are trying to find the functions $Y_n(y)$, it will be more convenient to write them as superposition of exponents rather than as superposition of hyperbolic functions (because $e^{-(\text{positive constant})\,y}$ tends to 0 while $e^{(\text{positive constant})\,y}$ tends to infinity as $y \to \infty$).

Additional problem 5. If $u = u(x,y)$, find the general solution of the partial differential equations

(a) $u_{xyy} = x\sin y$,

(b) $u_{xyy} = x\, e^{x^2}$.