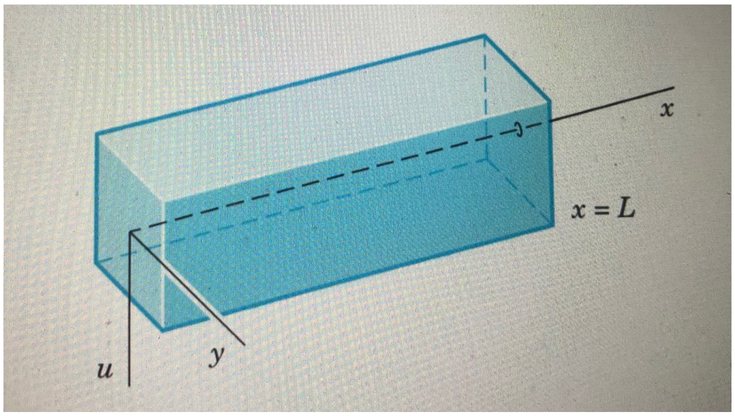
Math 3413.001: Physical Mathematics I

Homework 12, due by $\boxed{\text{NOON on Apr 22 (Wednesday)}}$

Lecture 25 (Apr 14) Due by NOON on 04/22/2020 : Section 9.6

In this homework, we will derive the solution for the motion of the vibrating elastic beam drawn below.



Arguing like in the case of vibrating string, we can deduce the partial differential equation satisfied by $u(x,t)$. It is an order 4 PDE given by

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4}, \qquad (*)$$

where $c^2 = EI/\rho A$ with $E =$ Young's modulus of elasticity, $I =$ moment of inertia of the cross section with respect to the y-axis in the figure above, $\rho =$ density, and $A =$ cross sectional area. Assume that $u(x,t) = F(x)G(t)$.

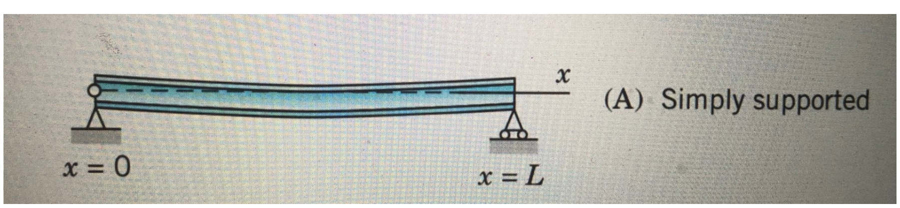
1. Substitute $u(x,t) = F(x)G(t)$ in the PDE $(*)$ to obtain the identity

$$\frac{F^{(4)}(x)}{F(x)} = -\frac{G''(t)}{c^2 G(t)} = -\lambda, \qquad \text{where } \lambda \text{ is a constant.}$$

   Use the above identity to obtain the two ODE's

$$F^{(4)}(x) + \lambda F(x) = 0, \qquad G''(t) - c^2 \lambda G(t) = 0.$$

2. The boundary conditions come from the assumption that the beam is simply supported as in the following figure.



They are

$$u(0,t) = u(L,t) = 0, \qquad u_{xx}(0,t) = u_{xx}(L,t) = 0.$$

The initial conditions are

$$u(x,0) = f(x) \qquad u_t(x,0) = 0.$$

Use these boundary and initial conditions, to get the two sets of ODEs.

$$F^{(4)}(x) + \lambda F(x) = 0, F(0) = F(L) = 0, F''(0) = F''(L) = 0,$$
$$G''(t) - c^2\lambda G(t) = 0, G'(0) = 0.$$

3. In this part, we will solve the ODE corresponding to $F(x)$:

$$F^{(4)}(x) + \lambda F(x) = 0, F(0) = F(L) = 0, F''(0) = F''(L) = 0.$$

It turns out that there are NO non-zero solutions if $\lambda = 0$ or $\lambda > 0$. Let $\lambda = -\beta^4 < 0$ for some positive real number β.

(a) Use methods from Chapter 3 to show that $F^{(4)}(x) - \beta^4 F(x) = 0$ has the solution

$$F(x) = A\cos(\beta x) + B\sin(\beta x) + Ce^{\beta x} + De^{-\beta x},$$

for unknown constants A, B, C, D.

(b) Use $F(0) = 0$ and $F''(0) = 0$ to derive

$$A = 0 \text{ and } C + D = 0.$$

(c) Use $F(L) = 0$ and $F''(L) = 0$ to derive

$$B\sin(\beta L) = 0 \text{ and } Ce^{\beta L} + De^{-\beta L} = 0.$$

Conclude that $C = D = 0$. Further conclude that all the non-zero solutions of the ODE are

$$F_n(x) = \sin(\frac{n\pi x}{L}), \text{ for } \beta = \frac{n\pi}{L}, \text{ which implies } \lambda = -(\frac{n\pi}{L})^4,$$

for $n = 1, 2, 3, \cdots$

4. In this part, we will find the corresponding solution for the ODE corresponding to $G(t)$:

$$G_n''(t) + c^2(\frac{n\pi}{L})^4 G_n(t) = 0, G_n'(0) = 0.$$

(a) Using methods from Chapter 3, show that

$$G_n(t) = K_n\cos(\frac{cn^2\pi^2 t}{L^2}) + M_n\sin(\frac{cn^2\pi^2 t}{L^2}).$$

(b) Use $G_n'(0) = 0$ to show that $M_n = 0$. Conclude that the non-zero solutions are

$$G_n(t) = \cos(\frac{cn^2\pi^2 t}{L^2}).$$

5. Write $u_n(x,t) = F_n(x)G_n(t)$ to conclude that

$$u(x,t) = \sum_{n=1}^{\infty} A_n u_n(x,t) = \sum_{n=1}^{\infty} A_n \cos(\frac{cn^2\pi^2 t}{L^2}) \sin(\frac{n\pi x}{L}).$$

Use the initial condition $u(x,0) = f(x)$ to conclude that A_n are the Fourier sine coefficients of $f(x)$.

Lecture 26 (Apr 16) Due by NOON on 04/22/2020 : Section 9.6

1. (**Plucked string**) A uniform string under tension and of length π is displaced at its center a distance π and released from initial velocity zero. Find its displacements as a function of time $y(x,t)$.
2. (**Struck string**) In contrast to a plucked string, the struck string is initially not displaced but has initial velocity imparted by a hammer applied at some segment. Suppose a hammer of width 2 strikes the string of length 20 at position $x = 10$ so that the initial velocity is

$$y_t(x,0) = \begin{cases} 10 & \text{if } 9 \le x \le 11; \\ 0 & \text{otherwise,} \end{cases}$$

starting from resting position $y(x,0) = 0$. Find its displacements as a function of time $y(x,t)$.
3. Solve the following boundary value problem

$$\begin{aligned} &9y_{tt} = y_{xx}, \qquad y(0,t) = y(5,t) = 0, \\ &y(x,0) = 4\sin(\pi x) - \sin(2\pi x) - 3\sin(5\pi x), \\ &y_t(x,0) = 3\sin(\frac{2\pi x}{5}) + 7\sin(\frac{4\pi x}{5}) - 2\sin(\frac{7\pi x}{5}). \end{aligned}$$

Suggested problems from the book (DO NOT SUBMIT): Pg 621-623, $\#1, 2, 4, 8, 10$