MATH 4433 Homework 12 Due 12/4/2014

Section 6.2: Exercises 17, 21. Hints and remarks:

- there is a useful hint in the book for Exercise 17;
- in Example 6.1.8 the formula $(x^n)' = nx^{n-1}$ was proved only for $n \in \mathbb{N}$ (for $n = 0$ the result is trivial), in Practice 6.1.9 the formula was generalized for arbitrary $n \in \mathbb{Z}$ by using the quotient rule, and in Practice 6.2.11 it was shown that $(x^{1/n})' = \frac{1}{n}x^{(1/n)-1}$ for $n \in \mathbb{N}$; in Exercise 21 you have to generalize this formula to the case of rational powers by writing $x^{m/n} = (x^{1/n})^m$ (with $m, n \in \mathbb{N}$) as a composition of functions.

Section 6.3: Exercises 3(f,h,j), 5, 6, 11. Hints and remarks:

- in part (f) of Exercise 8 apply l'Hospital's Rule to $\ln\left[(1+2x)^{1/x}\right] = \frac{\ln(1+2x)}{x}$; in part (h) merge the two terms in the difference into one before applying l'Hospital; in part (j) keep applying l'Hospital until the indeterminate form is resolved;
- Exercise 5 is very easy – check carefully if all conditions for applying l'Hospital's rule are satisfied;
- in Exercise 6 use directly the definitions for limit when $x \to c$ (for a finite $c \in \mathbb{R}$) and when $x \to \infty$: for example, assume that that $\lim_{x\to\infty} f(x) = L$, write down what this means (Definition 6.3.6), and rewrite it to prove that $\lim_{y\to 0^+} g(y) = L$ (the proof is literally a couple of lines, but please write it neatly in full detail); don't forget to prove also that $\lim_{y\to 0^+} g(y) = L$ implies $\lim_{x\to\infty} f(x) = L$;
- in Exercise 11, think of *simple* examples; for part (c), think how to use the function $\sin\frac{1}{x}$ in your example.

Section 6.4: Exercises 7, 13. Hints and remarks:

- [Important!] in Exercise 7, I want you to do a little more than what the problem you asking to do: please give a rigorous bound and exact bound on the error,

$$|R_n(x)| \le \frac{|x-x_0|^{n+1}}{(n+1)!} \sup_{c \text{ between } x_0 \text{ and } x} \left|f^{(n+1)}(c)\right| ,$$

  to find the error in approximating $\cos x$ by $p_5(x)$ on the interval $[0,1]$; then find the exact value of $\cos 1$ with a calculator and check if the true error, $|\cos 1 - p_5(1)|$ is within the rigorous error bound (it should be, of course);

- in Exercise 13, you will prove the spectacular result that the number e is irrational; follow the instructions in the statement of the problem – the proof is really simple.

  You are walking in the footsteps of the great – the irrationality of e was first proved by Euler in 1737, and the proof in the book follows the one given by Fourier in 1815! Congratulations!

Additional problem.

One can define different metics on function spaces, i.e., vector spaces of functions on which addition of two functions and multiplication of a number and a function are defined as usual: $(f+g)(x) := f(x)+g(x)$, $(c\,g)(x) := c\,g(x)$. Consider the vector space $\mathscr{V}$ of all functions $f:[0,1]\to\mathbb{R}$ for which $\int_0^1 |f(x)|^2\,\mathrm{d}x$ is finite:

$$\mathscr{V} := \left\{ f:[0,1]\to\mathbb{R} : \int_0^1 |f(x)|^2\,\mathrm{d}x < \infty \right\} .$$

(This is an infinitely-dimensional space!) On $\mathscr{V}$ define the metric $d_2 : \mathscr{V}\times\mathscr{V}\to\mathbb{R}$ as

$$d_2(f,g) := \sqrt{\int_0^1 |f(x)-g(x)|^2\,\mathrm{d}x}$$

(it can be shown that d_2 is indeed a metric). Consider the 2-dimensional subspace $\mathscr{L}$ of $\mathscr{V}$ that consists of all linear functions:

$$\mathscr{L} := \{\ell_{\alpha,\beta} : [0,1]\to\mathbb{R} : \ell_{\alpha,\beta}(x) = \alpha x+\beta\,,\ \alpha,\beta\in\mathbb{R}\} .$$

Assume that we want to approximate an arbitrary function $f\in\mathscr{V}$ by a function $\ell_{\alpha,\beta}\in\mathscr{L}$, where the parameters α and β are chosen to minimize the "distance" from f to $\ell_{\alpha,\beta}$. For "distance" we will use the metric d_2, so for a given function $f\in\mathscr{V}$, we want to find a linear function $\ell_{\alpha,\beta}\in\mathscr{L}$ that minimizes $d_2(f,\ell_{\alpha,\beta})$. Instead of minimizing $d_2(f,\ell_{\alpha,\beta})$, it is more convenient to minimize its square $d_2(f,\ell_{\alpha,\beta})^2$ – obviously, a function $\ell_{\alpha,\beta}$ that minimizes $d_2(f,\ell_{\alpha,\beta})^2$ will also minimize $d_2(f,\ell_{\alpha,\beta})$, and vice versa.

Let $f\in\mathscr{V}$ be the function $f(x)=x^2$. Find the function $\ell_{\alpha,\beta}\in\mathscr{L}$ that is closest to the function f, i.e., find the values of the parameters α and β such that the expression

$$d_2(f,\ell_{\alpha,\beta})^2 = \int_0^1 |x^2-\ell_{\alpha,\beta}(x)|^2\,\mathrm{d}x$$

has the smallest numerical value. You may use without deriving that

$$\int_0^1 \left(x^2-\alpha x-\beta\right)^2\,\mathrm{d}x = \frac{\alpha^2}{3}+\alpha\beta+\beta^2-\frac{\alpha}{2}-\frac{2\beta}{3}+\frac{1}{5} .$$

Food for Thought:

- Sec. **6.3**, exercises 1, 2, 13(a,b).