MATH 3423 Homework 2 Due Fri, 09/09/16

Before solving Problems 7 and 8 below, it may be useful to read the Food for Thought Problem at the end of this homework to refresh your memory on solving line integrals in Calculus.

Problem 1. Determine $\ln(-\mathrm{i})$ and $\mathrm{Ln}(-\mathrm{i})$.

Problem 2.

(a) Show that $\arcsin z = -\mathrm{i}\,\ln\left(\mathrm{i}z \pm \sqrt{1-z^2}\right)$.

Hint: Solve $\sin w = z$ (i.e., $\frac{\mathrm{e}^{\mathrm{i}w}-\mathrm{e}^{-\mathrm{i}w}}{2\mathrm{i}} = z$) for z. You can set $\xi := \mathrm{e}^{\mathrm{i}w}$, and rewrite $\frac{\mathrm{e}^{\mathrm{i}w}-\mathrm{e}^{-\mathrm{i}w}}{2\mathrm{i}} = z$ as a quadratic equation for ξ.

(b) Use your result from part (a) to solve the equation $\sin w = 2$. Note that this equation has no solution if w is real.

(c) Directly from the definition of the sine function, show that $\sin\left(\frac{\pi}{2} - \mathrm{i}\,\ln\left(2 \pm \sqrt{3}\right)\right) = 2$.

Problem 3. Evaluate $\sqrt{-1}^{\sqrt{-1}}$ (i.e., i^{i}).

Problem 4. Show that $\lim_{z\to 0} \frac{z^*}{z}$ does not exist by taking the limit along the ray $y = \alpha x$, where α is a real constant.

Problem 5. Classify all singularities of $f(z) = \frac{z}{(z^2+4)^2}$.

Problem 6. Let $u(x,y) = x^3 - 3xy^2$. Use Cauchy-Riemann equations to find a function $v(x,y)$ such that the function $f(z) = u(x,y) + \mathrm{i}\,v(x,y)$ is differentiable (where $z = x + \mathrm{i}y$).

Problem 7. Directly from the definition of a contour integral in $\mathbb{C}$, compute the value of the integral $\int_C f(x)\mathrm{d}z$, where $f(z) = x^2 - iy^2$, and C is the parabola $y = x^2$ between the points $(0,0)$ and $(3,9)$.

Problem 8. Let C be the circle of radius R centered at the origin, traversed in positive (i.e., counterclockwise) direction.

(a) Directly from the definition of a contour integral in $\mathbb{C}$, compute the value of integral $\oint_C z^*\,\mathrm{d}z$.

Hint: The most efficient way of parameterizing C is $z(\theta) = R\mathrm{e}^{\mathrm{i}\theta}$, where the parameter θ varies from 0 to 2π.

(b) Based solely on your result from part (a), can you say for sure whether the function z^* is analytic in the domain surrounded by C? Explain briefly how you came to your conclusion.

Problem 9. Use one of the results given on page 883 to give an upper bound for $\left|\int_C \frac{\mathrm{d}z}{z^3}\right|$, where C is the straight line segment from i to $3+5i$.

Problem 10. Evaluate $\oint_C \frac{z\,\mathrm{e}^{-z}}{z^3+8}\,\mathrm{d}z$, where C is the unit circle centered at the origin and traversed in positive direction.
Hint: Before you roll up your sleeves and start calculating, think! Or maybe think *instead of* rolling up your sleeves...

Food for Thought Problem – NOT TO BE TURNED IN!

Let $\mathbf{F}(x,y) = (2x^2+y)\,\mathbf{i} - 5xy\,\mathbf{j}$ be a vector field in $\mathbb{R}^2$. Evaluate the line integral $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ from $(0,0)$ to $(2,4)$ along the path C given by the parametric equations $\mathbf{r}(t) = (x(t),y(t)) = (t,t^2)$.

Hint: Recall that, by definition, $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,\mathrm{d}t$, where t_i and t_f are the initial and final values of the parameter t. We have $t_\mathrm{i}=0$, $t_\mathrm{f}=2$, $\mathbf{r}'(t) = \frac{\mathrm{d}}{\mathrm{d}t}(t,t^2) = \mathbf{i}+2t\,\mathbf{j}$, $\mathbf{F}(\mathbf{r}(t)) = 3t^2\,\mathbf{i} - 5t^3\,\mathbf{j}$, therefore

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,\mathrm{d}t = \int_0^2 \left(3t^2\,\mathbf{i} - 5t^3\,\mathbf{j}\right)\cdot(\mathbf{i}+2t\,\mathbf{j})\,\mathrm{d}t \\ &= \int_0^2 \left(3t^2 - 10t^4\right)\,\mathrm{d}t = \left(t^3 - 2t^5\right)\Big|_{t=0}^2 = 8 - 2\cdot 32 = -56\ .\end{aligned}$$