

Ross, Section 3.1: Exercise 3.3 (on page 19).

Please write your proof succinctly and neatly, pointing out at each step which axiom (or a previous part of Theorem 3.1 of Ross, or some of the Mini-Theorems proved in class) you are using. In referring to the axioms, please use the notations (A1)–(A4), (M1)–(M4), or (DL) from the handout that I gave in Lecture 3 (this handout is also linked from Lecture 3 on the class web-site).

Additional Problem 1. Write down the negation of each of the following statements. *No explanation is necessary!*

- (a) The function $f(x) = x^2 - 9$ is continuous at $x = 3$.
- (b) x is in A or x is not in B .
- (c) If $x < 7$, then $f(x)$ is not in C .
- (d) If the sequence $\{a_n\}$ is convergent, then $\{a_n\}$ is monotone and bounded.
- (e) Everyone likes Robert.
- (f) $\exists x$ in B such that $f(x) > 7$.
- (g) If $x > 5$, then $(f(x) < 3 \text{ or } f(x) > 7)$.

Additional Problem 2. Determine the truth value of each statement (T or F), assuming that x and y are real numbers. If the statement is true, explain why; if the statement is false, give a counterexample. *Note that this is only an exercise in elementary logic and notations, do not refer to the axioms for real numbers discussed in class!*

- (a) $\forall x$ and $y, \exists z$ such that $x + y = z$.
- (b) $\forall x \exists y$ such that $\forall z, x + y = z$.
- (c) $\exists x$ such that $\forall y \exists z$ such that $xz = y$.
- (d) $\forall x$ and $y, \exists z$ such that $yz = x$.

Additional Problem 3. We say that a number is *irrational* if it is not rational. Some of the statements below are true, others are false. Prove or give a counterexample to each of the statements below. You may use that $\sqrt{2}$ is an irrational number.

- (a) If x is irrational and y is irrational, then $x + y$ is irrational.

- (b) If $x + y$ is irrational, then x is irrational or y is irrational.

Hint: This statement is true. To prove it, give a proof of its *contrapositive*, i.e., assume that the negation of the statement (x is irrational or y is irrational) is true, and show that this implies the negation of the statement ($x + y$ is irrational).

- (c) If x is irrational and y is irrational, then xy is irrational.

- (d) If xy is irrational, then x is irrational or y is irrational.

Hint: See the Hint to part (b).

Additional Problem 4. Consider the function $f : [1, \infty) \rightarrow [0, \infty)$ defined by $f(x) = x^3 - x$.

- (a) Show that f is onto (i.e., surjective).

Hint: You may need to use the fact that $\lim_{x \rightarrow \infty} f(x) = \infty$ and the fact that f is continuous (so that the Intermediate Value Theorem holds).

- (b) Show that f is 1-to-1 (i.e., injective).

Hint: Use tools from Calculus to show that f is strictly increasing.

Additional Problem 5. Suppose that $f : A \rightarrow B$ is a function, and let C be a subset of A .

- (a) Give a counterexample to the following: $f(A \setminus C) \subseteq f(A) \setminus f(C)$.

Hint: The example $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$ may be useful.

- (b) Prove that $f(A) \setminus f(C) \subseteq f(A \setminus C)$.