MATH 4443 **Homework 2** **Due 2/1/18 (Thursday)**

Problem 1. Proving the Alternating Series Test (Theorem 2.7.7) amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - a_4 + \cdots \pm a_n$$

converges. Different characterizations of completeness lead to different proofs.

(a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.

(b) Prove the Alternating Series Test by using the Nested Interval Property (Theorem 1.4.1).

(c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the Monotone Convergence Theorem (for sequences) leads to a third proof of the Alternating Series Test.

Problem 2. Give an example of each or explain why the request is impossible referencing the relevant theorems.

(a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

(c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum (x_n + y_n)$ both converge, but $\sum y_n$ diverges.

(d) A sequence (x_n) satisfying $0 \leq x_n \leq \frac{1}{n}$, where $\sum (-1)^n x_n$ diverges.

Problem 3. Using the Cauchy Condensation Test and the basic facts about geometric series, prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Problem 4. Show that if $a_n > 0$ and $\lim(na_n) = L \neq 0$, then the series $\sum a_n$ diverges.

Problem 5. Given a series $\sum a_n$ with $a_n \neq 0$, the *Ratio Test* states that if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1 ,$$

then the series converges absolutely.

(a) Let the number s satisfy $r < s < 1$. Explain why there exists an N such that $n \geq N$ implies that $|a_{n+1}| \leq s|a_n|$.

(b) Why does $|a_N| \sum s^n$ converge?

(c) Show that $\sum |a_n|$ converges, and conclude that $\sum a_n$ converges.

Problem 6. *Abel's Test* for convergence states that if the series $\sum x_k$ converges, and if (y_k) is a bounded sequence satisfying

$$y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0 \ ,$$

then the series $\sum x_k y_k$ converges. Let s_n be the nth partial sum of the series $\sum x_k$.

(a) Use that $x_1 = s_1$ and $x_j = s_j - s_{j-1}$ for $j \geq 2$ to show that

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1}) \ .$$

(b) Use the Comparison Test to argue that $\sum s_k(y_k - y_{k+1})$ converges absolutely, and show how this leads directly to a proof of Abel's Test.

Food for Thought Problem 1.

(a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) by using the Cauchy Criterion for Series.

(b) Give another proof of the Comparison Test, this time by using the Monotone Convergence Theorem.

Food for Thought Problem 2. Determine whether each series converges or diverges. Justify your answer.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+2}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2^n)^3}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}$
8. $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n}$

10. $\sum_{n=1}^{\infty} n^{-1-1/n}$

11. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$

12. $\sum_{n=1}^{\infty} 2^n e^{-n}$

13. $\sum_{n=1}^{\infty} 3^n e^{-n}$

14. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

Food for Thought Problem 3. Determine whether each series converges absolutely, converges conditionally or diverges. Justify your answer.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$

2. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

3. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

4. $\sum_{n=1}^{\infty} \frac{(-5)^n}{2^n}$

5. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

6. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$

7. $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$