

Problem 1. [Thinking simply, again]

Find the exact value of the number γ defined by the following expression:

$$\gamma = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}} . \quad (1)$$

Hint: Look at the part of the right-hand side of (1) written with bold face digits. How it is related to γ ?

Problem 2. [Existence and uniqueness of roots of algebraic equations]

- (a) State the Intermediate Value Theorem (you can find it, for example, on page 89 of the 7th edition of Stewart's *Calculus*).
- (b) Consider the equation

$$f(x) = x - \cos x = 0 . \quad (2)$$

Prove that (2) has a solution in the interval $[0, \frac{\pi}{2}]$. Explain your reasoning in detail.

- (c) Prove that the solution of (2) in the interval $[0, \frac{\pi}{2}]$ is unique.

Hint: What can you say about the derivative of the function f on the interval $[0, \frac{\pi}{2}]$?

Problem 3. [Bifurcation in a logistic equation with constant harvesting]

Recall the logistic equation with constant harvesting,

$$\frac{dP}{dt} = \alpha P \left(1 - \frac{P}{K} \right) - H$$

where P is the population, t is the time, α is the reproduction rate of the species, K is the carrying capacity of the ecosystem, and H is the harvesting rate. After the non-dimensionalization $y := P/K$, $\tau := \alpha t$, $h := H/(\alpha K)$, the equation becomes

$$\frac{dy}{d\tau} = y(1 - y) - h . \quad (3)$$

Think of y and h as positive quantities.

- (a) Rewrite the condition $y(1 - y) - h = 0$ for a FP of (3) in the form $f(y) = g(y)$ with $f(y) = y(1 - y)$ and $g(y) = h$. Plot the graph of $f(y)$ for $y > 0$. What is the maximum value of f ? For which value of y is it attained?

- (b) Draw on the same plot the graphs of $f(y)$ and $g(y)$. How many fixed points of (3) are there for different values of h ? For which value h_0 of the parameter h does bifurcation occur? What is the value of the fixed point y_0^* at this value of h_0 ?
- (c) What kind of bifurcation occurs in the system (3)? Justify your answer by writing down the Taylor expansion of the right-hand side of (3) considered as a function of the variables y and h , in a Taylor series around (y_0^*, h_0) . What can you say about the stability of the fixed points?
- (d) Draw the bifurcation diagram of (3), i.e., the position of the fixed point(s) as a function of the parameter h . Write down the (implicit) equation of the curve on the bifurcation diagram. Don't forget to denote the positions of the stable FPs with a solid line, and the position of the unstable ones with a dashed line.

Problem 4. [Bifurcation in a logistic equation with linear harvesting]

Assume that the harvesting is a linear function of the population, i.e., consider the following modification of the logistic equation:

$$\frac{dy}{d\tau} = y(1 - y) - (a + by) , \quad (4)$$

where a and b are positive parameters.

- (a) Rewrite the condition $y(1 - y) - (a + by) = 0$ for a FP of (4) in the form $f(y) = g(y)$ with $f(y) = y(1 - y)$ and $g(y) = a + by$. Plot the graphs of $f(y)$ and $g(y)$ together, for three cases: when (4) has no FP, when (4) has exactly one FP, and when (4) has two FPs.
- (b) Write down the conditions for the equation (4) to have exactly one FP. Solve them to obtain a relation between the parameters a and b .
Hint: Recall that the graphs of $f(y)$ and $g(y)$ must “touch” at a point, which gives you two conditions.
- (c) Plot the relation obtained in part (b) in the (a, b) plane, and indicate how many FPs of (4) are there in each region in your plot.