

**Problem 1. [Thinking simply, again]**

Find the exact value of the number  $\gamma$  defined by the following expression:

$$\gamma = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}} . \quad (1)$$

*Hint:* Look at the part of the right-hand side of (1) written with bold face digits. How it is related to  $\gamma$ ?

**Problem 2. [Existence and uniqueness of roots of algebraic equations]**

(a) State the Intermediate Value Theorem (you can find it, for example, on page 89 of the 7th edition of Stewart's *Calculus*).

(b) Consider the equation

$$f(x) = x - \cos x = 0 . \quad (2)$$

Prove that (2) has a solution in the interval  $[0, \frac{\pi}{2}]$ . Explain your reasoning in detail.

(c) Prove that the solution of (2) in the interval  $[0, \frac{\pi}{2}]$  is unique.

*Hint:* What can you say about the derivative of the function  $f$  on the interval  $[0, \frac{\pi}{2}]$ ?

**Problem 3. [Bifurcation in a logistic equation with constant harvesting]**

Recall the logistic equation with constant harvesting,

$$\frac{dP}{dt} = \alpha P \left( 1 - \frac{P}{K} \right) - H$$

where  $P$  is the population,  $t$  is the time,  $\alpha$  is the reproduction rate of the species,  $K$  is the carrying capacity of the ecosystem, and  $H$  is the harvesting rate. After the non-dimensionalization  $y := P/K$ ,  $\tau := \alpha t$ ,  $h := H/(\alpha K)$ , the equation becomes

$$\frac{dy}{d\tau} = y(1 - y) - h . \quad (3)$$

Think of  $y$  and  $h$  as positive quantities.

(a) Rewrite the condition  $y(1 - y) - h = 0$  for a FP of (3) in the form  $f(y) = g(y)$  with  $f(y) = y(1 - y)$  and  $g(y) = h$ . Plot the graph of  $f(y)$  for  $y > 0$ . What is the maximum value of  $f$ ? For which value of  $y$  is it attained?

- (b) Draw on the same plot the graphs of  $f(y)$  and  $g(y)$ . How many fixed points of (3) are there for different values of  $h$ ? For which value  $h_0$  of the parameter  $h$  does bifurcation occur? What is the value of the fixed point  $y_0^*$  at this value of  $h_0$ ?
- (c) What kind of bifurcation occurs in the system (3)? Justify your answer by writing down the Taylor expansion of the right-hand side of (3) considered as a function of the variables  $y$  and  $h$ , in a Taylor series around  $(y_0^*, h_0)$ . What can you say about the stability of the fixed points?
- (d) Draw the bifurcation diagram of (3), i.e., the position of the fixed point(s) as a function of the parameter  $h$ . Write down the (implicit) equation of the curve on the bifurcation diagram. Don't forget to denote the positions of the stable FPs with a solid line, and the position of the unstable ones with a dashed line.

**Problem 4. [Bifurcation in a logistic equation with linear harvesting]**

Assume that the harvesting is a linear function of the population, i.e., consider the following modification of the logistic equation:

$$\frac{dy}{d\tau} = y(1 - y) - (a + by) , \quad (4)$$

where  $a$  and  $b$  are positive parameters.

- (a) Rewrite the condition  $y(1 - y) - (a + by) = 0$  for a FP of (4) in the form  $f(y) = g(y)$  with  $f(y) = y(1 - y)$  and  $g(y) = a + by$ . Plot the graphs of  $f(y)$  and  $g(y)$  together, for three cases: when (4) has no FP, when (4) has exactly one FP, and when (4) has two FPs.
- (b) Write down the conditions for the equation (4) to have exactly one FP. Solve them to obtain a relation between the parameters  $a$  and  $b$ .  
*Hint:* Recall that the graphs of  $f(y)$  and  $g(y)$  must “touch” at a point, which gives you two conditions.
- (c) Plot the relation obtained in part (b) in the  $(a, b)$  plane, and indicate how many FPs of (4) are there in each region in your plot.