MATH 5163 Homework 2 Due Thu, 1/30/14

Problem 1. [Differentiation of an integral depending on a parameter]
The solution of the initial value problem for the wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\ , \qquad x \in \mathbb{R}\ , \quad t \ge 0\ ,$$
$$u(x,0) = 0\ ,$$
$$u_t(x,0) = h(x)$$

(where c is a positive constant) is given by a particular case of the so-called *D'Alembert's formula*,

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} h(z)\,\mathrm{d}z \tag{1}$$

(you do not need to prove this).
The following formula holds for differentiating an integral whose limits and integrand depend on some parameter α:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{\phi(\alpha)}^{\psi(\alpha)} F(y,\alpha)\,\mathrm{d}y = F(\psi(\alpha),\alpha)\,\psi'(\alpha) - F(\phi(\alpha),\alpha)\,\phi'(\alpha) + \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial F}{\partial \alpha}(y,\alpha)\,\mathrm{d}y \tag{2}$$

(again, there is no need to prove this).

(a) Use the above formula for differentiating an integral depending on parameter to find $u_t(x,t)$ and $u_x(x,t)$ (where $u(x,t)$ is given by (1)).

(b) Use the above formula for differentiating an integral depending on parameter to find $u_{tt}(x,t)$ and $u_{xx}(x,t)$ and check that these derivatives satisfy the wave equation.

(c) Check that the expression for $u(x,t)$ given by the D'Alembert's formula satisfies the initial conditions $u(x,0) = 0$ and $u_t(x,0) = h(x)$.

Problem 2. [Compatibility between the PDE and the Neumann BC]

(a) Use some of Green's formulas to show that

$$\int_U f\,\mathrm{d}V = \oint_{\partial U} g\,\mathrm{d}S \tag{3}$$

is a necessary condition for existence of a solution of the Neumann boundary value problem for the Poisson equation

$$\begin{aligned} &\Delta u = f \qquad \text{in } U\ ,\\ &\left.\frac{\partial u}{\partial \nu}\right|_{\partial U} = g\ . \end{aligned} \tag{4}$$

Here $\dfrac{\partial u}{\partial \nu}(\mathbf{x})$ is the directional derivative of u in the direction of the outward unit normal vector $\boldsymbol{\nu}(\mathbf{x})$ at the point $\mathbf{x} \in \partial U$.

(b) Recall that the steady-state temperature u satisfies the stationary heat equation,

$$k\,\Delta u = -\Psi \ ,$$

where the meaning of k and Ψ is recalled in the Hint below. The Neumann problem for this equation means that the heat flux through the boundary is controlled, i.e., that we control the normal derivative, $\dfrac{\partial u}{\partial \nu} = \boldsymbol{\nu} \cdot \nabla u$ at the boundary ∂U. Give a physical interpretation in terms of stationary temperature distribution of the necessary condition (3) for existence of a solution of the Neumann boundary value problem (4) for the stationary heat equation.

Hint: Recall that propagation of heat obeys Fourier's law, $\mathbf{j} = -k\nabla u$, where k is a positive constant characterizing the medium, $\mathbf{j}$ is the flux density of the heat flow, and u is the temperature, so that the amount of heat energy leaving the domain U per unit time is $\oint_{\partial U} \mathbf{j} \cdot \mathrm{d}\mathbf{S}$. Use the Divergence Theorem to rewrite this as a volume integral over U. On the other hand, if Ψ is the volume density of the power of heat sources inside the domain U, then the amount of heat generated per unit time is $\int_U \Psi \,\mathrm{d}V$.

Problem 3. [Uniqueness of solutions of Robin BVP for the Poisson equation]
Prove that there is at most one solution of the Robin BVP for the Poisson equation,

$$\begin{aligned} &\Delta u = f \qquad \text{in } U \ , \\ &\left(\frac{\partial u}{\partial \nu} + a(\mathbf{x})\, u\right)\Big|_{\partial U} = g \ . \end{aligned} \tag{5}$$

in $C^2(U) \cap C(\bar{U})$. Here $a : \partial U \to R$ is a strictly positive function. The Robin boundary condition has a clear physical meaning if u stands for the steady-state temperature: it corresponds to a convective heat exchange at the boundary of the domain U. The positivity of the function a corresponds to the fact that the heat energy goes from hot to cold places.
Hint: Recall the proof of the uniqueness of solutions of Dirichlet BVP from Lecture 2.

Problem 4. [Inverse Mean Value Theorem for harmonic functions]
Prove the Inverse Mean Value Theorem for harmonic functions: if $u \in C^2(U) \cap C(\bar{U})$ satisfies the mean value property, i.e., if

$$u(\mathbf{x}) = \fint_{\partial B(\mathbf{x},r)} u(\mathbf{y})\,\mathrm{d}S_{\mathbf{y}} \tag{6}$$

for each ball $B(\mathbf{x}, r) \subset U$, then u is harmonic in U.

Hint: Recall the function $\Phi : (0, r] \to \mathbb{R}$ defined as the average of u over the sphere $\partial B(\mathbf{x}, t)$:

$$\Phi(t) = \fint_{\partial B(\mathbf{x},t)} u(\mathbf{y})\, \mathrm{d}S_{\mathbf{y}} \tag{7}$$

(in class we also wrote the derivative of this function and related it with an integral of Δu). What does the property (6) imply about Φ'? Rereading the proof of the Mean Value Theorem for harmonic functions from Lecture 3 will be helpful.

Problem 5. [Subharmonic functions]

A function $u \in C^2(U) \cap C(\bar{U})$ is said to be *subharmonic* if

$$\Delta u \geq 0 \qquad \text{in } U\ . \tag{8}$$

In this problem you will prove several properties of subharmonic functions. The reasoning should be very similar to the arguments in the proofs of the Mean Value Theorem for harmonic functions (Lecture 3) and the Strong Maximum Principle (Lecture 4).

(a) Prove that, if u is subharmonic, then

$$u(\mathbf{x}) \leq \fint_{\partial B(\mathbf{x},r)} u(\mathbf{y})\, \mathrm{d}S_{\mathbf{y}} \tag{9}$$

for any ball $B(\mathbf{x}, r) \subset U$.

Hint: Use (8) to show that the function Φ is defined in (7) is nondecreasing on $(0, r]$, which, together with the continuity of Φ on $[0, r]$ will imply the desired inequality.

(b) Show that (9) implies

$$u(\mathbf{x}) \leq \fint_{B(\mathbf{x},r)} u(\mathbf{y})\, \mathrm{d}V_{\mathbf{y}}\ .$$

(c) Let $\mathbf{x}_0 \in U$ be such that $M := u(\mathbf{x}_0) = \max_{\mathbf{x} \in \bar{U}} u(\mathbf{x})$, and $B(\mathbf{x}_0, r)$ be a ball that lies entirely in U. Explain why we can conclude that $u(\mathbf{x}) = M$ for every $\mathbf{x} \in B(\mathbf{x}_0, r)$.

(d) Use part (c) to prove that, if u is subharmonic, then

$$\max_{\mathbf{x} \in \bar{U}} u(\mathbf{x}) = \max_{\mathbf{x} \in \partial U} u(\mathbf{x})\ .$$

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function, i.e., $f''(t) \geq 0$ for any $t \in \mathbb{R}$, and let u be a harmonic function in U. Show that

$$\Delta(f \circ u)(\mathbf{x}) = f''(u(\mathbf{x}))\, |\nabla u(\mathbf{x})|^2 + f'(u(\mathbf{x}))\, \Delta u(\mathbf{x})\ ,$$

and use this to show that the composition $f \circ u : U \to \mathbb{R}$ is a subharmonic function.