# MATH 5403 Homework 2 Due Tue, 10/15/19

**Problem 1. [Broken extremals, Weierstrass-Erdman conditions]**
In this problem you will find the broken extremals (i.e., piecewise $C^1$ extremals) of the functional

$$J[y] = \int_0^2 \left[(y')^4 - 6(y')^2\right] \mathrm{d}x \,, \tag{1}$$

subject to the boundary conditions $y(0) = 0$, $y(2) = 0$, assuming that $y'$ may have one discontinuity point at $x = c$.

(a) Write down the Euler-Lagrange equations for (1), and explain why the solutions are only straight lines.

(b) Let the solution $y$ to the left and to the right of $c$ be given by the functions

$$\begin{aligned} y_-(x) &= \alpha x + \beta \,, \qquad x \in [0, c) \,, \\ y_+(x) &= \gamma x + \delta \,, \qquad x \in [c, 2] \,. \end{aligned}$$

Impose the boundary conditions to determine the constants $\beta$ and $\delta$.

(c) The continuity of $y$ imposes a relation between the constants $\alpha$ and $\gamma$ (which also involves the still unknown abscissa $c$ of the break point).

(d) Impose the Weierstrass-Erdman conditions

$$\begin{aligned} (F - y'F_{y'})\big|_{x=c^-} &= (F - y'F_{y'})\big|_{x=c^+} \,, \\ F_{y'}\big|_{x=c^-} &= F_{y'}\big|_{x=c^+} \,, \end{aligned}$$

and show that they can be written as

$$(\alpha - \gamma)(\alpha + \gamma)(\alpha^2 + \gamma^2 - 2) = 0 \,, \tag{2}$$

$$(\alpha - \gamma)(\alpha^2 + \alpha\gamma + \gamma^2 - 3) = 0 \,. \tag{3}$$

(e) Note that (2) yields three possibilities:

$$\text{(i) } \alpha = \gamma \,, \qquad \text{(ii) } \alpha = -\gamma \,, \qquad \text{(iii) } \alpha^2 - \gamma^2 = 2 \,.$$

First consider only Case (i). Is (3) satisfied in this case? What is the value of $c$? Write down the solution $y$ in this case. Is it a broken extremal?

(f) Consider Case (ii); is the solution a broken extremal?

(g) Consider Case (iii). Discuss the nature of the extremal you obtain.

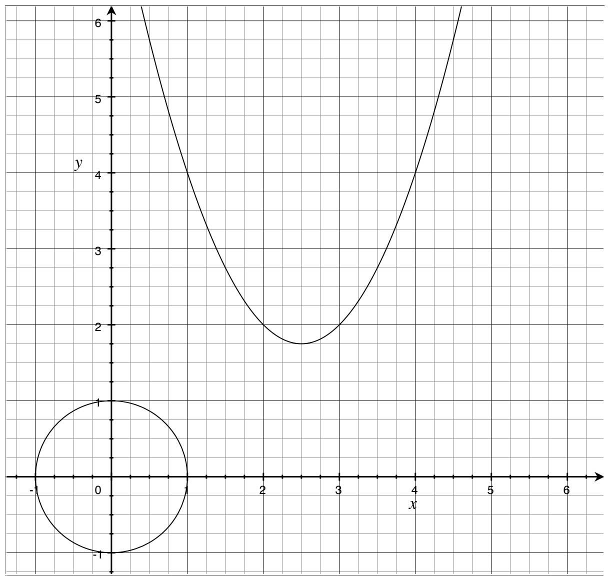


Figure 1: The unit circle $x^2 + y^2 = 1$ and the parabola $y = x^2 - 5x + 8$.

**Problem 2. [Movable end points, transversality conditions]**
Consider the circle $x^2 + y^2 = 1$ and the parabola $y = \psi(x) = x^2 - 5x + 8$ in $\mathbb{R}^2$, drawn in Figure 1. Your goal in this problem is to find the minimum distance between these two curves by using the methods of Calculus of Variations.

(a) Let $U$ be an interval in $\mathbb{R}$, $y : U \to \mathbb{R}$ be a $\mathrm{C}^1$ function, and $\{(x, y(x)) : x \in U\}$ be the graph of $y$. Let $x_0, x_1 \in U$, and set $y_0 = y(x_0)$ and $y_1 = y(x_1)$. Write the functional $J[y]$ giving the length of the graph of $y$ between the points $(x_0, y_0)$ and $(x_1, y_1)$.

(b) Write down and solve the Euler-Lagrange equation for the action functional $J[y]$.

*Hint:* You have already solved a more general version of this problem in Homework 1. The solution is geometrically obvious without doing any math.

(c) It is clear from the figure that the point on the unit circle that is closest to the parabola will be in the upper half of the unit circle, so let us assume that the left end of the distance minimizing curve is on the curve $y = \phi(x) = \sqrt{1 - x^2}$, $x \in (-1, 1)$.

Let $(x_0, y_0)$ and $(x_1, y_1)$ be the end points of the curve minimizing the distance between the graphs of $\phi$ and $\psi$; clearly, $y_0 = \phi(x_0)$ and $y_1 = \psi(x_1)$. Write down the transversality conditions for the distance-minimizing curve connecting the points $(x_0, y_0)$ and $(x_1, y_1)$; use the concrete expressions for the action minimizer $y$ found in part (b) and the functions $\phi$ and $\psi$.

(d) Solve the transversality conditions written in part (c) and whatever other equations you need to find the points $(x_0, y_0)$ and $(x_1, y_1)$ and the minimizer $y$ connecting them.

(e) Find the numerical value of the distance between the graphs of $\phi$ and $\psi$.

**Problem 3. [A membrane hanging in the Earth's gravity field]**

Consider a loop made of metal wire; let the loop be in the $(x, y)$-plane (i.e., $z = 0$) in the 3-dimensional space. Consider a membrane (or, equivalently, a soap film) whose end is attached to the wire. If there were no gravity, the equilibrium position of the membrane will be in the plane $z = 0$ because of the surface tension. The gravity force changes the equilibrium shape of the membrane. We will determine this shape as a minimization problem of an action functional.

Let the domain in the $(x, y)$-plane that is surrounded by the wire be denoted by $D$. Let the function $u(x, y, t)$ describe the position of the membrane at time $t$, i.e., let the equation of the membrane at time $t$ be $z = u(x, y, t)$. Define the following notations:

- $\rho$ is the area density of the mass of the membrane (unit: $\mathrm{kg/m^2}$);
- $\tau$ is the surface tension (unit: $\mathrm{N/m} = \mathrm{kg/s^2}$);
- $f(x, y, t)$ is the area density of the external forces, i.e., force per unit area of the membrane (unit: $\mathrm{N/m} = \mathrm{kg/(m\,s^2)}$); the area density of gravity force is $f = -\rho g$, where the minus sign reflects the fact the that gravity acceleration $\mathbf{g}$ points downward.

In the case when the unknown function $u$ depends on more than one variable, the situation is the following. Let $u$ be a function depending on time $t$ and on the spatial coordinate(s) $\mathbf{r}$; in this particular problem $\mathbf{r}$ stands for $(x, y) \in \mathbb{R}^2$. The action for the function $u(t, \mathbf{r})$ is

$$I[u] = \int_{t_1}^{t_2} \iint_D \mathscr{L}\big(u, \nabla u, u_t, \mathbf{r}, t\big)\, \mathrm{d}A\, \mathrm{d}t \ ,$$

where $D$ is the given domain in $(x, y)$-plane (surrounded by the wire), and $\mathrm{d}A = \mathrm{d}x\, \mathrm{d}y$ is the area element in the $(x, y)$-plane. The function

$$\mathscr{L}\big(u, \nabla u, u_t, \mathbf{r}, t\big) := \mathscr{L}\big(u, u_x, u_y, u_t, x, y, t\big)$$

is called *Lagrangian density*. The Euler-Lagrange equation in this case is

$$\frac{\partial \mathscr{L}}{\partial u} - \nabla \cdot \left( \frac{\partial \mathscr{L}}{\partial \nabla u} \right) - \frac{\partial}{\partial t} \left( \frac{\partial \mathscr{L}}{\partial u_t} \right) = 0 \ ,$$

i.e.,

$$\frac{\partial \mathscr{L}}{\partial u} - \frac{\partial}{\partial x} \left( \frac{\partial \mathscr{L}}{\partial u_x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial \mathscr{L}}{\partial u_y} \right) - \frac{\partial}{\partial t} \left( \frac{\partial \mathscr{L}}{\partial u_t} \right) = 0 \ .$$

The Lagrangian density is equal to the difference of the area density of the kinetic energy and the area density of the potential energy. The area density of the kinetic energy is $\mathscr{T} = \dfrac{\rho u_t^2}{2}$; the area density of the gravitational potential energy is $\mathscr{U}_{\text{grav}} = \rho g u$.

The total potential energy due to the surface tension is equal to the surface tension times the change of the area of the membrane due to the shape of the membrane. The area of the membrane when it is flat is $A(D)$ (recall that $D$ is the domain the $(x,y)$-plane surrounded by the wire), and its area at time $t$ is given by (look into any Calculus book)

$$\iint_D \sqrt{1+|\nabla u|^2}\,\mathrm{d}A = \iint_D \left(1+|\nabla u|^2\right)^{1/2}\,\mathrm{d}x\,\mathrm{d}y \approx \iint_D \left(1+\tfrac{1}{2}|\nabla u|^2\right)\,\mathrm{d}x\,\mathrm{d}y$$

where $|\nabla u|^2 = u_x^2 + u_y^2$. Here we used the fact that, and for $|\xi|<1$ and $\alpha \notin \{0,1,2,3,\ldots\}$,

$$(1+\xi)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} \xi^k \ ,$$

where

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} \ .$$

This gives us that the total elastic potential energy (due to the surface tension) is

$$\tau\left[\iint_D \left(1+|\nabla u|^2\right)^{1/2}\,\mathrm{d}x\,\mathrm{d}y - (\text{area of } D)\right] \approx \frac{\tau}{2}\iint_D \left(u_x^2+u_y^2\right)\,\mathrm{d}x\,\mathrm{d}y \ .$$

Therefore the area density of the elastic potential energy is

$$\mathscr{U}_{\text{elastic}} = \frac{\tau}{2}\left(u_x^2+u_y^2\right) \ .$$

Putting everything together, we obtain

$$\mathscr{L}\big(u,u_x,u_y,u_t,x,y,t\big) = \mathscr{T} - \left(\mathscr{U}_{\text{grav}} + \mathscr{U}_{\text{elastic}}\right) = \frac{\rho u_t^2}{2} - \rho g u - \frac{\tau}{2}\left(u_x^2+u_y^2\right) \ . \qquad (4)$$

(a) Derive the Euler-Lagrange equation corresponding to the Lagrangian density (4). In writing your final result, use the Laplacian operator $\Delta$ defined by $\Delta u = u_{xx} + u_{yy}$. The equation you will obtain can be written in the form

$$\Delta u - \frac{1}{c^2}\,u_{tt} = \frac{\rho g}{\tau} \ ,$$

where $c$ is the speed of the propagation of the waves in the membrane. What is $c$ in the equation you derived?

(b) Now assume that the rim of the membrane (i.e., the wire on which the membrane is suspended) is circular with radius $R$. Consider the static equilibrium of the membrane, when the function $u$ does not depend on $t$. Because of the symmetry of the system, it is clear that in polar coordinates $(r,\theta)$ the shape of the membrane will depend only on $R$, so that we can set $u(x,y,t) = U(r)$, where $r = \sqrt{x^2+y^2}$. A tedious calculation (you do *not* need to do it here!) shows that in polar coordinates the Laplacian is

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \ .$$

Use this to show that the function $U(r)$ satisfies the boundary-value problem

$$\begin{aligned} &\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\left[r\,U'(r)\right] = \frac{\rho g}{\tau}\ , \qquad r \in [0, R]\ , \\ &U(R) = 0\ , \quad |U(r)| < \infty \end{aligned} \tag{5}$$

(the condition $|U(r)| < \infty$ is imposed for obvious physical reasons).

(c) Solve the boundary value problem (5) for $U(r)$ from (5).

(d) At which point will the membrane hang most? Give physical reasons for your answer. Write down an expression for the maximum hanging of the membrane. How does it depend on the radius $R$?

**Problem 4. [Incorporating resistance forces in the Lagrangian]**
The motion of a membrane in a viscous fluid is governed by the equation

$$\rho\, u_{tt} = \tau\, \Delta u - \gamma u_t + f\ . \tag{6}$$

The notations have the following meaning:

- $z = u(x, y, t)$ is the function describing the position of the membrane at time $t$;
- $\rho$ is the area density of the mass of the membrane (unit: $\mathrm{kg/m^2}$);
- $\tau$ is the surface tension (unit: $\mathrm{kg/s^2}$);
- $\gamma$ is the coefficient of resistance (unit: $\mathrm{kg/(m^2 s)}$);
- $f(x, y, t)$ is the area density of the external forces, i.e., force per unit area of the membrane; for example, the gravity force will give $f = -\rho g$ (unit: $\mathrm{kg/(m\,s^2)}$).

This system is dissipative, i.e., the energy is not conserved (because of the term containing the velocity $u_t$), and cannot be described directly by a Lagrangian density. However, it can be formally derived from the Lagrangian density

$$\mathscr{L}(u, u_t, u_x, u_y, x, y, t) = \left[\frac{\rho}{2}\, u_t^2 - \frac{\tau}{2}\, |\nabla u|^2 + f(x, y, t)\, u\right] \mathrm{e}^{\frac{\gamma}{\rho}\, t}\ . \tag{7}$$

Perform the calculations to derive (6) as the Euler-Lagrange equation corresponding to (7).