MATH 5463 Homework 2 Due Thu, Feb 12

Problems 28, 31, 37 from Section 3.5 of the book.

Hint to Problem 28(a): Use Proposition 3.13(a).

Additional problem 1. Let (x, y) and (r, θ) be the Cartesian and the polar coordinates in the plane $\mathbb{R}^2$, respectively, and let $\mathbf{0}$ be the origin, $(x, y) = (0, 0)$. For $n \in \mathbb{N}$ define the numbers

$$a_n = \frac{1}{10^n}\left(1 - \frac{1}{10^n}\right) , \qquad b_n = \frac{1}{10^n}$$

and the sets $D_n = \{(x, y) \in \mathbb{R}^2 : r \in [a_n, b_n)\}$. Each D_n is an *annulus* (plural: *annuli*), i.e., the area between two concentric circles: $D_n = B(b_n, \mathbf{0}) \setminus B(a_n, \mathbf{0})$. Note that D_n are disjoint. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$f(x, y) = \sum_{n \in \mathbb{N}} \chi_{D_n}(x, y) ,$$

i.e., $f(x, y)$ is 1 if (x, y) belongs to some annulus D_n and 0 otherwise. Note that $f(\mathbf{0}) = 0$.

(a) Is the function f continuous at $\mathbf{0}$?

(b) Show that the Lebesgue measure (i.e., the area) of D_n is smaller than $\dfrac{2\pi}{10^{3n}}$.

(c) Let $(x, y) \in D_n$, i.e., $r \in [a_n, b_n)$. Prove that the average $(A_r f)(\mathbf{0})$ of f over the ball $B(r, \mathbf{0})$ decreases with n as $\dfrac{C}{10^n}$ for some constant $C > 0$.

Solution:

$$\begin{aligned}
(A_r f)(\mathbf{0}) &= \frac{1}{m(B(r,\mathbf{0}))} \int_{B(r,\mathbf{0})} f \, \mathrm{d}m \ \le\ \frac{1}{m(B(a_n,\mathbf{0}))} \int_{B(b_n,\mathbf{0})} f \, \mathrm{d}m \\
&= \frac{1}{\pi a_n^2} \sum_{j=n}^{\infty} m(D_n) \ <\ \frac{10^{2n}}{\pi \left(1 - \frac{1}{10^{2n}}\right)^2} \sum_{j=n}^{\infty} \frac{2\pi}{10^{3j}} \\
&\le \frac{2 \cdot 10^{2n}}{\pi} \frac{2\pi}{10^{3n}} \sum_{k=0}^{\infty} \frac{1}{10^{3k}} \ =\ \frac{C}{10^n} .
\end{aligned}$$

(d) Let (x, y) be a point in the area between D_n and D_{n+1}, i.e, let the distance from (x, y) to $\mathbf{0}$ be $r \in [b_{n+1}, a_n)$. Give an upper bound on the average $(A_r f)(\mathbf{0})$ of f over the ball $B(r, \mathbf{0})$ in terms of n similarly to the bound in part (c).

(e) Based on the bounds in parts (c) and (d), what can you conclude about the behavior of the averages $(A_r f)(\mathbf{0})$ as $r \to 0$? How about the Hardy-Littlewood maximal function $(Hf)(\mathbf{0})$? Is the point $\mathbf{0}$ in the Lebesgue set of f?

Food for thought. Problems 27, 29 from Section 3.5 of the book.

Food for thought. Read Examples 1, 2, 14, 15, 16, 17 from Chapter 8 of *Counterexamples in Analysis* by Gelbaum and Olmsted, and think about the properties of the Cantor function and the measure associated with it.