MATH 2513 **Homework 3** **Due 2/7/19 (Thursday)**

Please explain briefly but clearly your reasoning (unless it is totally obvious from your answer, i.e., when you have to list the elements of a set or to draw a set in the plane).

Please write the problems in the same order as they are given in the assignment.

Note that the odd-numbered problems have answers at the end of Hammack's book. I strongly suggest that you do *all* odd-numbered problems for practice; moreover, many of them are very similar to the assigned homework problems.

Hammack, Section 2.7: Exercises 2, 4, 6, 8, 10.

Hammack, Section 2.9: Exercises 2, 6, 10.

Hammack, Section 2.10: Exercises 2, 4, 6, 8, 10.

Additional problem 1. Write the negation of each statement.

(a) Everyone likes Robert.

(b) Some students work part-time.

(c) There exists an element x of the set B for which $f(x) > k$.

(d) If $x > 5$, then $f(x) \leq 3$ or $f(x) > 7$.

Additional problem 2. Determine the truth value of each statement, assuming that x, y, and z are real numbers. Provide a brief (but clear) explanation or a counterexample.

(a) $\forall x$ and y, $\exists z$, $x + y = z$.

(b) $\forall x$, $\exists y$ s.t. $\forall z$, $x + y = z$.

(c) $\exists x$ s.t. $\forall y$, $\exists z$, $xy = z$.

(d) $\forall x$ and y, $\exists z$, $yz = x$.

(e) $\forall x$ and y, $\exists z$ such that $z > y$ implies that $z > x + y$.

Additional problem 3. Consider the following

> **Definition.** A function is said to be *periodic* if there exists a $k > 0$ such that for every $x \in \mathbb{R}$, $f(x + k) = f(x)$.

(a) Rewrite the above definition in mathematical notation by using $\forall$, $\exists$, and $\Rightarrow$, as appropriate.

(b) Write the negation of part (a) by using the same symbolism.

Additional problem 4.

> **Definition.** A function $f : A \to B$ is said to be *injective* if for every x and y in A, if then $f(x) = f(y)$, then $x = y$.

(a) Rewrite the above definition in mathematical notation by using $\forall$, $\exists$, and $\Rightarrow$, as appropriate.

(b) Write the negation of part (a) by using the same symbolism.

(c) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is *not* injective. Explain briefly.

Food for Thought problem 1. [1] Determine the truth value of the following statement, assuming that x, y, and z are real numbers.

$$\forall x, \exists y \text{ s.t. } \forall z, [(z > y) \Rightarrow (z > x + y)] \,. \tag{1}$$

Justify your reasoning.

Solution: The statement (1) is False. To show this, we can prove that its negation is True. The negation of the statement (1) is

$$\exists x \text{ s.t. } \forall y, \exists z, \sim [(z > y) \Rightarrow (z > x + y)] \,. \tag{2}$$

Recalling that $\sim (P \Rightarrow Q)$ is equivalent to (i.e., has the same truth value as) $P \vee (\sim Q)$, and using the De Morgan's Laws, we can restate the negation (2) of the statement (1) as

$$\exists x \text{ s.t. } \forall y, \exists z, [(z > y) \wedge (z \leq x + y)] \,. \tag{3}$$

To show that the negation (3) of the original statement (1) is True, we have to find a value of x such that for any y we can find a value z such that $z > y$ and $z \leq x + y$. These two inequalities can be rewriten as $y < z \leq x + y$. Clearly, if we take any $x > 0$, we can find a value of z that satisfies these inequalities for every y, namely, we can take any $z \in (y, x + y]$. So, take, e.g., $x = 2$, and for any value of y take, e.g., $z = y + 1 \in (y, 2 + y]$. For these values of x, y, and z, the statement $[(z > y) \wedge (z \leq x + y)]$ (see (3)) is True. Therefore, the values (namely: choose $x = 2$, y arbitrary, for these values of x and y choose $z = y + 1$) provide a counterexample to the original statement (1). Going back to the original statement (1), we see that $z > y$ does *not* imply $z > x + y$ for *any* strictly positive x.

Here is a simpler explanation why (1) is False. Take, say, $x = 2$. Given any y, let $z = y + 1$. Then "$z > y$" is True, but "$z > x + y$" is False.

[1] "Foot for Thought" problems are for you to think about, but they do *not* need to be turned in with the regular homework.