MATH 3423 Homework 3 Due Mon, 9/16/13

Problem 1. The solution of the initial value problem for the wave equation

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\ , \qquad x \in \mathbb{R}\ , \quad t \geq 0\ ,$$
$$u(x,0) = 0\ ,$$
$$u_t(x,0) = h(x)$$

(where v is a positive constant) is given by a particular case of the so-called *D'Alembert's formula*,

$$u(x,t) = \frac{1}{2v}\int_{x-vt}^{x+vt} h(z)\,\mathrm{d}z \tag{1}$$

(you do not need to prove this).

The following formula holds for differentiating an integral whose limits and integrand depend on some parameter α:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{\phi(\alpha)}^{\psi(\alpha)} F(y,\alpha)\,\mathrm{d}y = F(\psi(\alpha),\alpha)\,\psi'(\alpha) - F(\phi(\alpha),\alpha)\,\phi'(\alpha) + \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial F}{\partial \alpha}(y,\alpha)\,\mathrm{d}y \tag{2}$$

(again, there is no need to prove this).

(a) Use the above formula for differentiating an integral depending on parameter to find $u_t(x,t)$ and $u_x(x,t)$ (where $u(x,t)$ is given by (1)).

(b) Use the above formula for differentiating an integral depending on parameter to find $u_{tt}(x,t)$ and $u_{xx}(x,t)$ and check that these derivatives satisfy the wave equation.

(c) Check that the expression for $u(x,t)$ given by the D'Alembert's formula satisfies the initial conditions $u(x,0) = 0$ and $u_t(x,0) = h(x)$.

Problem 2. Recall the fact that $\int_{-\infty}^{\infty} \mathrm{e}^{-y^2}\,\mathrm{d}y = \sqrt{\pi}$. By change of variables in the integral, one can easily obtain that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha y^2}\,\mathrm{d}y = \sqrt{\frac{\pi}{\alpha}}\ . \tag{3}$$

Below you will obtain some facts about integrals of the form $\int_{-\infty}^{\infty} y^n\,\mathrm{e}^{-y^2}\,\mathrm{d}y$, for some $n \in \mathbb{N}$.

(a) Explain why $\int_{-\infty}^{\infty} y^{2n-1}\,\mathrm{e}^{-y^2}\,\mathrm{d}y = 0$ for any $n \in \mathbb{N}$.

Hint: The reason is *very* simple!

(b) Use formula (2) from Problem 1 to differentiate both parts of equation (3) with respect to the parameter α, and then set $\alpha = 1$, to obtain an expression for $\int_{-\infty}^{\infty} y^2\, \mathrm{e}^{-y^2}\, \mathrm{d}y$.

(c) Similarly to part (b), find $\int_{-\infty}^{\infty} y^4\, \mathrm{e}^{-y^2}\, \mathrm{d}y$.

Problem 3. Find the four 4th roots of i.

Problem 4. Starting with $\int_0^{\infty} \mathrm{e}^{-\beta^2 y^2}\, \mathrm{d}y = \frac{\sqrt{\pi}}{2\beta}$, let $\beta = \frac{1-\mathrm{i}}{\sqrt{2}}$ (and notice that $\beta^2 = -\mathrm{i}$) to show that $\int_0^{\infty} \cos(x^2)\, \mathrm{d}x = \int_0^{\infty} \sin(x^2)\, \mathrm{d}x = \sqrt{\frac{\pi}{8}}$.

Problem 5. Directly from the definition of $\sinh z$, show that the Taylor expansion of $\sinh z$ around 0 is $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$.

Problem 6. Determine $\ln(-\mathrm{i})$ and $\mathrm{Ln}(-\mathrm{i})$.

Problem 7.

(a) Show that $\arcsin z = -\mathrm{i}\, \ln\left(\mathrm{i}z \pm \sqrt{1-z^2}\right)$.

Hint: Solve $\sin w = z$ (i.e., $\frac{\mathrm{e}^{\mathrm{i}w} - \mathrm{e}^{-\mathrm{i}w}}{2\mathrm{i}} = z$) for z. You can set $\xi := \mathrm{e}^{\mathrm{i}w}$, and rewrite $\frac{\mathrm{e}^{\mathrm{i}w} - \mathrm{e}^{-\mathrm{i}w}}{2\mathrm{i}} = z$ as a quadratic equation for ξ.

(b) Use your result from part (a) to solve the equation $\sin w = 2$. Note that this equation has no solution if w is real.

(c) Directly from the definition of the sine function, show that $\sin\left(\frac{\pi}{2} - \mathrm{i}\, \ln\left(2 \pm \sqrt{3}\right)\right) = 2$.

Problem 8. Evaluate $\sqrt{-1}^{\sqrt{-1}}$ (i.e., i^{i}).

Problem 9. Show that $\lim_{z \to 0} \frac{z^*}{z}$ does not exist by taking the limit along the ray $y = mx$, where m is a real constant.

Problem 10. Classify all singularities of $f(z) = \frac{z}{(z^2+4)^2}$.

Problem 11. Let $u(x,y) = x^3 - 3xy^2$. Use Cauchy-Riemann equations to find a function $v(x,y)$ such that the function $f(z) = u(x,y) + \mathrm{i}\, v(x,y)$ is differentiable (where $z = x + \mathrm{i}y$).