MATH 3423 Homework 3 Due Thu, 09/12/19

Problem 1. Give direct proofs of the following identities:

(a) $(z_1 z_2)^* = z_1^* \, z_2^*$;

(b) $\left(\dfrac{1}{z}\right)^* = \dfrac{1}{z^*}$;

(c) $|z_1 z_2| = |z_1| \, |z_2|$.

Problem 2. In the complex plane, describe in words and sketch the domain D given by the inequalities

$$1 < |z + \mathrm{i}| \leq 3 \ .$$

Denote the boundaries that do not belong to D with dashed lines, and the boundaries that belong to D with solid lines.

Problem 3. Evaluate $\mathrm{Re}\, \dfrac{5 + \mathrm{i}}{2 - \mathrm{i}}$.

Problem 4. Express the following function $w(z) = \dfrac{z^*}{z}$ in the form $w(z) = u(x, y) + \mathrm{i}\, v(x, y)$ (where u and v are real-valued functions of two real variables).

Problem 5. Determine the modulus and the principal argument of the complex numbers

(a) $2 - 2\mathrm{i}$;

(b) $-\mathrm{i}$;

(c) $-3 + \sqrt{3}\,\mathrm{i}$.

Hint: Drawing pictures in a problem like this is very helpful.

Problem 6. Express the following complex numbers in Cartesian coordinates (i.e., in the form $z = x + \mathrm{i}y$):

(a) $\mathrm{e}^{\ln 2 - (\pi/4)\mathrm{i}}$;

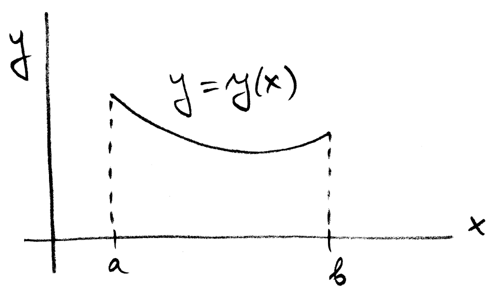
(b) $6\, \mathrm{e}^{2\pi \mathrm{i}/3}$.

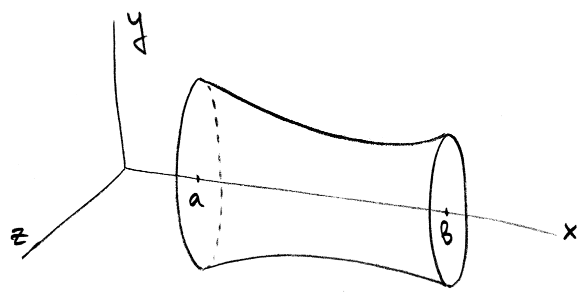
Problem 7. Evaluate $(\sqrt{3}+\mathrm{i})^{14}$; write your result in the form $(\sqrt{3}+\mathrm{i})^{14} = x + \mathrm{i}y$.
Hint: First write $\sqrt{3}+\mathrm{i}$ in polar coordinates.

Problem 8. Starting with $\displaystyle\int_0^\infty \mathrm{e}^{-\beta^2 y^2}\,\mathrm{d}y = \frac{\sqrt{\pi}}{2\beta}$, let $\beta = \dfrac{1-\mathrm{i}}{\sqrt{2}}$ (and notice that $\beta^2 = -\mathrm{i}$) to show that

$$\int_0^\infty \cos\left(x^2\right)\mathrm{d}x = \int_0^\infty \sin\left(x^2\right)\mathrm{d}x = \sqrt{\frac{\pi}{8}}\ .$$

Problem 9. Consider the surface of revolution obtained by rotating a curve $y = y(x)$ in the (x, y)-plane around the x-axis, for $x \in [a, b]$, as shown in the figure below.





Let S stand for the area swept by the curve (note that we do only consider the area of the "cylindrical" part of the figure in the picture on the right, not the areas of the two flat circles at $x = a$ and $x = b$).
The goal in this problem is to find the curve $y = y(x)$ that produces a surface of revolution with the smallest possible area if the two endpoints $(a, y(a))$ and $(b, y(b))$ in the figure on the left are given. This problem has a simple physical interpretation – a soap film whose ends are attached to the two "hoops" at $x = a$ and at $x = b$ in the figure on the right will take exactly the shape that minimizes the surface area because of the surface tension (in this problem we neglect the effect of gravity).

(a) Write down the expression for the area S of the surface of revolution. This expression is a functional of the function y of the form

$$S[y] = \int_a^b L(y, y', x)\,\mathrm{d}x\ , \qquad y' := \frac{\mathrm{d}y}{\mathrm{d}x}\ . \tag{1}$$

Hint: You have solved this problem in Calculus (see Stewart's *Calculus*, 7 ed., Sec. 8.2).

(b) Write down the Euler-Lagrange equations for the functional S written in (1).

(c) Note that the function $L(y, y', x)$ does not depend explicitly on x, which according to Problem 1 of Homework 2 implies that the quantity $y'\,\dfrac{\partial L}{\partial y'} - L$ should be a constant;

set

$$y' \frac{\partial L}{\partial y'} - L = \text{const} = 2\pi C_1 \ . \tag{2}$$

Write (2) explicitly and show that it implies that

$$y' = \pm \frac{1}{C_1} \sqrt{y^2 - C_1^2} \ ; \tag{3}$$

assume that $C_1 \neq 0$ (because $C_1 = 0$ simply imply that $y(x) \equiv 0$).

(d) **[Food for thought only, not to be turned in!]**

Show that the general solution of the ODE (3) can be written in the form

$$y(x) = C_1 \cosh \frac{x - C_2}{C_1} \ . \tag{4}$$

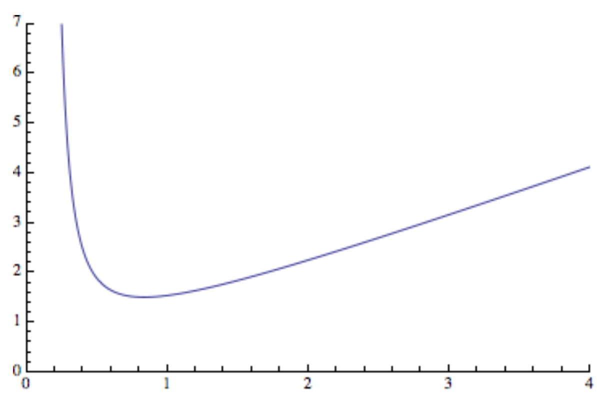
Hint: The integral $\displaystyle\int \frac{\mathrm{d}y}{\sqrt{y^2 - C_1^2}}$ can be solved by using the substitution $y = C_1 \cosh \xi$ and recalling that $\cosh^2 \xi - \sinh^2 \xi = 1$ and $(\cosh \xi)' = \sinh \xi$.

(e) **[Food for thought only, not to be turned in!]**

Now let the initial and the final values of x be $a = -1$ and $b = 1$, and impose the boundary conditions $y(-1) = \beta = y(1)$. From the left-right symmetry of the problem, it is clear that $y(x)$ must be an even function, and since cosh is an even function with a single minimum, it is clear from (4) that the solution $y(x)$ must have the form $y(x) = C_1 \cosh \dfrac{x}{C_1}$. Imposing the remaining condition,

$$\beta = y(1) = C_1 \cosh \frac{1}{C_1} \ , \tag{5}$$

however, poses a problem: this equation for C_1 has no solution for β in a certain range because the right-hand side of (5) (as a function of C_1) is shown in the figure.



Clearly, if β is smaller than the minimum value of the right-hand side (which is about 1.5, achieved for C_1 approximately equal to 0.83), then there is no value of C_1 that satisfies (5). The absence of solution is due to the fact that the function $y(x)$ that minimizes the area S is not continuous – namely, the function minimizing S is the discontinuous function

$$y(x) = \begin{cases} \beta & x = \pm 1 \ , \\ 0 & x \in (-1, 1) \ . \end{cases}$$

For more on this, see, e.g.,

H. Sagan. *Introduction to the Calculus of Variations.*
Dover Publications, 1992, Section 2.6.