MATH 4073 Homework 3 Due Fri, 09/17/10

Problem 1. Consider the equation

$$f(x) = x - \cos x = 0 \ . \tag{1}$$

(a) Prove that the equation (1) has a solution in the interval $[0, \frac{\pi}{2}]$. Please specify which theorem you used to come to this conclusion.

(b) Prove that the solution of (1) in the interval $[0, \frac{\pi}{2}]$ is unique.

(c) Use the Matlab code `bisection.m` (available at the class web-site, together with instructions how to run it) to find the root of (1) in $[0, \frac{\pi}{2}]$. Use tolerance 10^{-12} and run the code verbosely, so that you can see the results at each step. Please attached your printout.

(d) If E_n is the error in the nth step of the bisection method, then one can write $E_n = \mathcal{O}(\beta_n)$ for some (simple) sequence $\{\beta_n\}$. What is β_n for the bisection method? Explain why theoretically, and then from your numerical results in part (c).

Problem 2. Consider the function $g(x) = -x^3 + 6x^2 - 11x + 8$. The Mathematica command

```
Plot[{ - x^3 + 6*x^2 - 11*x + 8, x}, {x,1.0,3.0}]
```

would display the graphs of g and the diagonal $y = x$ on for x in the interval $[1, 3]$ (there is no need to attach a printout).

(a) Show (by hand) that $x = 2$ is a fixed point of the function g.

(b) Compute the values of $g(\frac{3}{2})$ and $g(\frac{5}{2})$, and check that $\frac{3}{2} < g(\frac{3}{2}) < g(\frac{5}{2}) < \frac{5}{2}$. What can you say about $g([\frac{3}{2}, \frac{5}{2}])$ (i.e., about the interval of values that $g(x)$ takes when x traverses the whole interval $[\frac{3}{2}, \frac{5}{2}]$)? What can you conclude from this about the existence of a fixed point of g in the interval $[\frac{3}{2}, \frac{5}{2}]$? Which theorem have you used?

(c) Explain why you cannot apply Theorem 2.2(b) to show that the fixed point in the interval $[\frac{3}{2}, \frac{5}{2}]$ is unique.

(d) Since you could not use Theorem 2.2(b) to show the uniqueness of the fixed point $x = 2$ of the function g, try something else. Define the function $f(x) := g(x) - x$. Show that f is strictly decreasing everywhere (even at $x = 2$), and use this fact to prove that f cannot have more than one zero.

Hint: Show that the first derivative of f can be written as $-3(x - 2)^2$.

(e) Use the Matlab program `fixedpoint.m` (available at the class web-site) fo find the fixed point of g with tolerances `tol` $= 10^{-2}$, 10^{-3}, 10^{-4}, 10^{-5}, and 10^{-6}, with initial value $p_0 = 1.5$. To see the number of iterations the code will perform, set the variable `verbose` to be equal to 1. Make sure that the parameter `nmax` that you pass to the program (the maximum number of iterations allowed) is large enough. To see more digits of the results, type `format long`. The stopping criterion this program uses is $|p_n - p_{n-1}| <$ `tol`. The Matlab command

```
fixedpoint(inline('-x^3+6*x^2-11*x+8'), 1.5, 1e-2, 100000, 1)
```

produces the value given in the table below, after 9 iterations. Display your results in a table:

Desired tolerance	Value obtained	Number of iterations
10^{-2}	1.800656708346558	9
$\vdots$	$\vdots$	$\vdots$

Look at the computed values of the fixed point. Do they look correct within the desired tolerance?

Remark: To get help about a particular Matlab command, say, about `inline`, type `help inline` in Matlab.

(f) In your opinion, why did the program need such a large number of iterations before the stopping criterion was met and the program stopped? (And recall that the precision obtained was far from the desired tolerance).

Hint: Look at Corollary 2.4 (page 59). Why doesn't it work in our problem?

Problem 3. Consider the one-parameter family of functions

$$g_a(x) = ax(1 - x) \ . \tag{2}$$

Here the real number a is a parameter; in this problem we will assume that $a > 1$.

(a) Find all fixed points of the recursion relation $p_n = g_a(p_{n-1})$. One of them does not depend on a, and is not very interesting. Show that if $a > 1$, the other fixed point is strictly positive; let p stand for this fixed point.

(b) In this and the next several parts of the problem you will study the behavior of the iterates of g_a. For this purpose you may use the following Mathematica code:

```
p = 0.2;
a = 2.8;
g[x_] = a * x * (1-x);
For[ i = 1, i <= 200, i++,
  { p = g[p],
    Print[ i, "     ", p],
  }
]
```

For this and the following parts of this problem, please do NOT attach the printouts, just describe what you observe!

Run this code with $a = 2.8$ (and $p_0 = 0.2$). Do the iterates p_n tend to a limit? What is the numerical value of this limit? Compare it with your theoretical prediction from part (a).

(c) Now run the code with $a = 3.3$ (again with $p_0 = 0.2$). Do the iterates tend to a limit? Look closely at the last iterates (with $n \approx 200$) – what do you observe?

(d) Run the code with $a = 3.5$ (again with $p_0 = 0.2$). Again, look closely at the last iterates (hint: look at p_{196} and p_{200}).

(e) Run the code with $a = 3.55$ (again with $p_0 = 0.2$). Again, describe the asymptotic behavior of the sequence $\{p_n\}_{n=0}^{\infty}$ and describe what you see (look at p_{192} and p_{200}).

It turns out that, if the parameter a keeps growing, at some values of a the asymptotic behavior of the iterates of g_a changes abruptly – the terminology is that at these values g_a undergoes *period-doubling bifurcations*. The discovery in mid-1970s of some striking properties of this infinite sequence of bifurcations by Mitchell Feigenbaum (a physicist, back then at the Los Alamos National Laboratory) led to a rapid development of the modern *Theory of Dynamical Systems* (which studies the asymptotic behavior of high iterates of maps). A famous early article on simple ecological models that exhibit interesting phenomena is "Simple mathematical models with very complicated dynamics" by Robert May (published in *Nature* **261** (1976), 459–467), which is attached to my e-mail with this homework. It is a pleasure to read – take a look at it when you have time.