MATH 4193/5103 Homework 3 Due 3/3/17 (Tuesday)

Problem 1. [Linearization, change of variables, and (un)stable manifolds]
Consider the system

$$\begin{aligned} x' &= 1 - x + y^4 \ , \\ y' &= y \ . \end{aligned} \tag{1}$$

(a) Show that $\mathbf{x}_* = (1, 0)$ is the only fixed point of (1).

(b) Linearize the system (1) around the fixed point $(1, 0)$.

(c) Write down the linearized system $\mathbf{u}' = D\mathbf{f}(\mathbf{x}_*)\,\mathbf{u}$ and solve it to find the function $\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ if $\mathbf{u}(0) = \mathbf{u}^{(0)} = \begin{pmatrix} u_1^{(0)} \\ u_2^{(0)} \end{pmatrix}$. From the solution $\mathbf{u}(t)$, would you say that $\mathbf{x}_* = (1, 0)$ is a stable FP or that it is unstable one? Explain briefly your reasoning.

Hint: Solving the linearized system is very simple because you can solve the equations for $u_1(t)$ and $u_2(t)$ separately.

Remark: Figure 1 shows the integral lines of (1) on the square $(x, y) \in [-3, 3] \times [-3, 3]$ on the left, and the zoom $(x, y) \in [0.4, 1.6] \times [-0.6, 1.6]$ on the right. The solution $\mathbf{u}(t)$ of the linearized system looks very similar to the zoomed plot (except that the plot of $\mathbf{u}(t)$ is centered at $(0, 0)$.

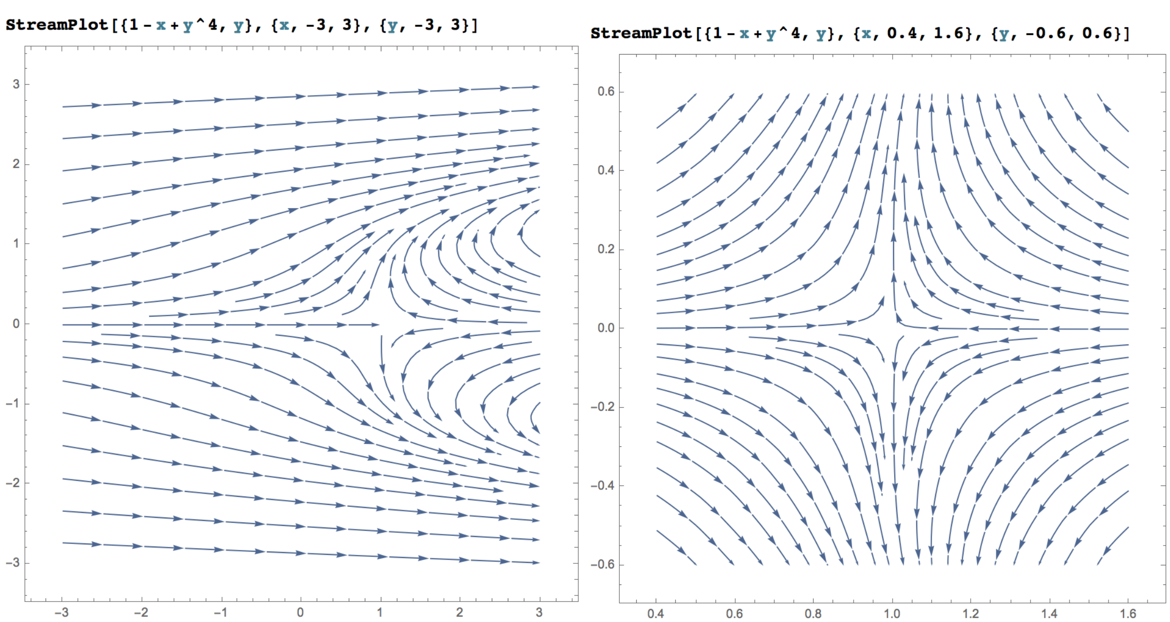

Figure 1: Integral lines of the system (1) and a zoom near the fixed point $\mathbf{x}_* = (1, 0)$.

(d) Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be a given function and $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ be an autonomous system of ODEs for the unknown function $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$. Let $\mathbf{x}_* \in \mathbb{R}^n$ be a FP of this system, i.e., $\mathbf{f}(\mathbf{x}_*) = \mathbf{0}$. The *stable manifold* $W^{\mathrm{s}}_{\mathbf{x}_*}$ of the fixed point $\mathbf{x}_*$ is the set of points $\mathbf{x}^{(0)} \in \mathbb{R}^n$ such that the solution $\boldsymbol{\phi}_t(\mathbf{x}^{(0)})$ of the system of ODEs with initial condition $\mathbf{x}^{(0)}$ tends to $\mathbf{x}_*$ as $t \to \infty$, i.e.,

$$W^{\mathrm{s}}_{\mathbf{x}_*} := \left\{ \mathbf{x}^{(0)} \in \mathbb{R}^n \, : \, \lim_{t\to\infty} \boldsymbol{\phi}_t(\mathbf{x}^{(0)}) = \mathbf{x}_* \right\} .$$

The solution of the system (1) with initial conditions $(x(0), y(0)) = (x_0, y_0)$ is

$$\begin{aligned} x(t) &= 1 + \left(x_0 - \tfrac{y_0^4}{5} - 1 \right) \mathrm{e}^{-t} + \tfrac{y_0^4}{5} \, \mathrm{e}^{4t} \, , \\ y(t) &= y_0 \, \mathrm{e}^{t} \, ; \end{aligned} \tag{2}$$

you do *not* need to prove this! Use (2) to show that the stable manifold $W^{\mathrm{s}}_{\mathbf{x}_*}$ of the fixed point $\mathbf{x}_* = (1, 0)$ of the system (1) is the x-axis, i.e., $W^{\mathrm{s}}_{\mathbf{x}_*} = \{(x_0, 0) : x_0 \in \mathbb{R}\}$.

(e) In the notations of part (d), the *unstable manifold* $W^{\mathrm{u}}_{\mathbf{x}_*}$ of the fixed point $\mathbf{x}_*$ is the set of points $\mathbf{x}^{(0)} \in \mathbb{R}^n$ such that the solution $\boldsymbol{\phi}_t(\mathbf{x}^{(0)})$ of the system of ODEs with initial condition $\mathbf{x}^{(0)}$ tends to $\mathbf{x}_*$ as $t \to -\infty$, i.e.,

$$W^{\mathrm{u}}_{\mathbf{x}_*} := \left\{ \mathbf{x}^{(0)} \in \mathbb{R}^n \, : \, \lim_{t\to-\infty} \boldsymbol{\phi}_t(\mathbf{x}^{(0)}) = \mathbf{x}_* \right\} .$$

Use (2) to find the unstable manifold $W^{\mathrm{u}}_{\mathbf{x}_*}$ of the fixed point $\mathbf{x}_* = (1, 0)$ of the system (1).

(f) Sometimes it is possible to change variables in a neighborhood of the fixed point so that the original ODE (1) looks exactly like the linearized system in the new variables. In this particular example, it is possible to do this. Define new functions, $\widetilde{x}(t)$ and $\widetilde{y}(t)$, related to $x(t)$ and $y(t)$ by

$$\widetilde{x}(t) = x(t) - \frac{1}{5} \, y(t)^4 - 1 \, , \qquad \widetilde{y}(t) = y(t) \, .$$

Directly from (1), obtain a system of first-order ODEs that $\widetilde{x}(t)$ and $\widetilde{y}(t)$ satisfy.

Problem 2. [Saddle-node and subcritical pitchfork in a 1-parameter family]

(a) Consider the 3-parameter family of first-order ODEs

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay + by^3 - cy^5 \, , \tag{3}$$

where a, b, and c are parameters, satisfying $b > 0$, $c > 0$. Two of the three parameters can be eliminated by the change of variables

$$y = \sqrt{\frac{b}{c}} \, x \, , \qquad t = \frac{c}{b^2} \, \tau \, .$$

Show that after this change of variables, the 3-parameter family (3) is transformed into the 1-parameter family

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = f_\mu(x) := \mu x + x^3 - x^5 \ . \tag{4}$$

Express the parameter μ in terms of the parameters a, b, and c in the original equation (3).

(b) The fixed points of the 1-parameter family (4) are the roots of the algebraic equation $f_\mu(x) = 0$, which can be written as

$$f_\mu(x) = -x(x^4 - x^2 - \mu) = 0 \ .$$

It is clear that $x_0^* = 0$ is always a root of this equation, i.e., a fixed point of (4). The other fixed points of (4) are the roots of the biquadratic equation

$$x^4 - x^2 - \mu = 0 \ ,$$

and are given by

$$x_{1,2,3,4}^* = \pm\sqrt{\frac{1 \pm \sqrt{1+4\mu}}{2}} \ . \tag{5}$$

The graph of the fixed points x^* of (4) as functions of the parameter μ is shown in Figure 2 (the stability of the fixed points is not indicated). In the rest of this problem

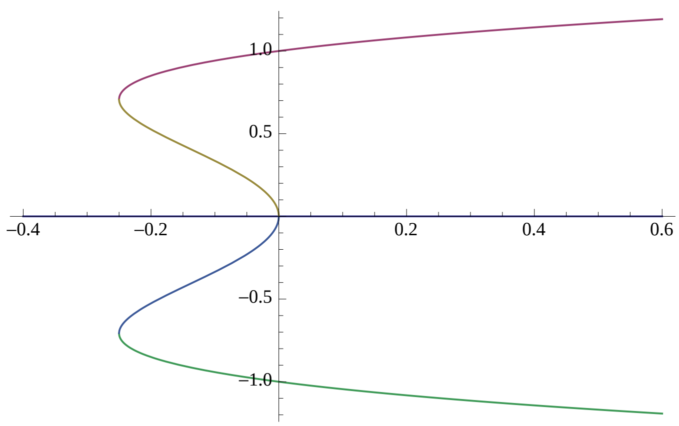

Figure 2: Bifurcation diagram of the system (4): plot of the position of the fixed points x_j^* as functions of the parameter μ.

you will identify all the bifurcations that occur in (4).

Using (5), show that for $\boxed{\mu > 0}$, the ODE (4) has exactly three FPs: one FP equal to 0, and the other two are "far from 0" (they are close to -1 and 1 for small positive values of μ). Write the expressions for two nonzero FPs (you can recognize them from (5)).

Hint: You can answer this question very easily if you think about the following: what is the sign of $f_\mu'(0)$ (for $\mu > 0$), and how do $f_\mu(x)$ behave when $x \to -\infty$ and when $x \to \infty$?

(c) For $\boxed{\mu > 0}$, find $f'_\mu(0)$, sketch roughly the graph of $f_\mu(x) = \mu x + x^3 - x^5$ (explain why it looks like this), and indicate the stability of all three fixed points as usual (full circles for the stable FPs and empty circles for the unstable FPs); there is no need to do long computations, just look at the graph.

(d) Using (5), show that for $\boxed{-\frac{1}{4} < \mu < 0}$, the ODE (4) has five FPs: one FP equal to 0, two "far from 0" FPs (near -1 and 1 for small values of μ), and two "near 0" FPs (approaching to 0 when $\mu \to 0^-$). Write the exact expressions for all FPs in this case.

(e) For $\boxed{-\frac{1}{4} < \mu < 0}$, find $f'_\mu(0)$ and use this information (and the general form of $f_\mu(x)$) to draw a rough sketch of $f_\mu(x)$. From your sketch, draw the "direction of motion" and indicate the stability of each FP as usual. What kind of bifurcation does the system undergo at $\mu = 0$?

(f) For $\boxed{\mu < -\frac{1}{4}}$, explain why there is only one fixed point of (4); draw a rough sketch of $f_\mu(x)$, and indicate the stability of this fixed point.

(g) What kind of bifurcation does the system exhibit at $\mu = -\frac{1}{4}$? You can answer this question by just looking at the graphs of $f_\mu(x)$ drawn in the previous part of the problem.

(h) The Taylor expansion of the function $h(z) = (1+z)^\beta$ for β not equal to a non-negative integer is

$$(1+z)^\beta = 1 + \beta z + \frac{\beta(\beta-1)}{2!}z^2 + \frac{\beta(\beta-1)(\beta-2)}{3!}z^3 + \cdots \approx 1 + \beta z \ ;$$

in particular,

$$\sqrt{1+z} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \frac{5}{128}z^4 + \cdots \ .$$

Use this for $\mu = -\varepsilon$, where ε is a very small positive number, to show that the FPs near 0 are approximately $\pm\sqrt{-\mu}$.

(i) In the (μ, x^*)-plane, sketch the bifurcation diagram of the ODE (4) (drawn in Figure 2), put all the important values on the axes, and indicate the stabilities of all fixed points (solid lines for the stable and dashed lines for the unstable FPs).

(j) **[Only if you are taking the class as 5103!]**

Write down the Taylor expansion of x^* near the point $(\mu, x^*) = (-\frac{1}{4}, \frac{1}{\sqrt{2}})$ in the bifurcation diagram, as a function of the parameter μ for $\mu = -\frac{1}{4} + \varepsilon$, where ε is a very small positive number.