MATH 4193/5103 Homework 3 Due Thu, Mar 14, 2013

Problem 1. [Dynamics on a circle]

Consider the interval $[-\pi, \pi]$ (with its ends identified) as a model of the circle S. Define the two-parameter family of functions $f : S \to \mathbb{R}$ by

$$f(\theta) = \omega - a + \frac{a}{\pi}|\theta| \; . \tag{1}$$

This function is piece-wise linear (i.e., its graph consists of segments of straight lines), and satisfies $f(-\pi) = f(\pi) = \omega$, $f(0) = \omega - a$; see Figure 1. For simplicity, in all parts of this problem assume that $\boxed{\omega \geq 0}$.

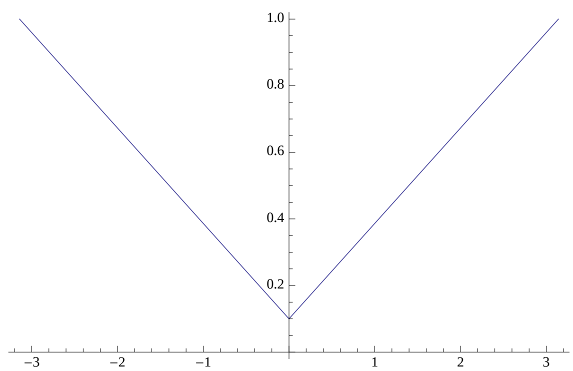

Figure 1: The graph of the function (1) for $\omega = 1$ and $a = 0.9$.

Consider the system

$$\dot{\theta} = f(\theta) \; , \tag{2}$$

where $f : S \to \mathbb{R}$ is the function defined in (1), and $\theta : \mathbb{R} \to S$ is an unknown function.

(a) For each value of $\omega \geq 0$, find an explicit expression for the value of a for which the system (2) undergoes a bifurcation. What kind of bifurcation is it?

(b) For a given value of $\omega \geq 0$, and for a value of a in the range in which the system (2) has exactly two fixed points, θ_1^* and θ_2^* (assume that $\theta_1^* < \theta_2^*$) find the values of θ_1^* and θ_2^* expressed in terms of the values of ω and a. Plot these values in the (a, θ^*) plane for a given value of $\omega \geq 0$. Indicate the value of ω in the (a, θ^*) plane. Use a solid line to denote the stable fixed point and a dashed line to denote the unstable fixed point in the (a, θ^*) plane.

(c) In the (ω, a) plane, indicate the region in which the system (2) has two fixed points, and the region where it has no fixed points.

(d) For given values of $\omega \geq 0$ and a such that the system (2) has no fixed points, find the period T as a function of ω and a. For a given $\omega \geq 0$, sketch the graph of T vs. a.

Problem 2. [Determinant-trace plane]

Consider the following one-parameter family of linear systems of ordinary differential equations:

$$\dot{\mathbf{x}} = \mathsf{A}\mathbf{x} \,, \qquad \text{where} \qquad \mathsf{A} := \begin{pmatrix} a & \frac{1+a}{2} \\ \frac{1-a}{2} & 0 \end{pmatrix} \,. \tag{3}$$

(a) Express the determinant Δ and the trace τ of the matrix A as functions of the parameter a and sketch the path traced out by this family of linear systems in the (Δ, τ)-plane as a varies. What is the equation of this path? Find the values of Δ and τ where this path intersects the Δ and the τ axes, respectively. For which values of a do these intersections occur?

(b) Discuss the bifurcations that occur along this path, and compute the values of a where these bifurcations occur. In other words, identify the values of a at which the behavior of the solutions of the linear system (3) changes dramatically.

Hint: There are two such values, $a_1 < 0 < a_2$. What are a_1 and a_2?

(c) For each of the following three cases:

Case I: $a < a_1$,

Case II: $a_1 < a < a_2$,

Case III: $a_2 < a$,

do the following:

(c_1) find the eigenvalues and eigenvectors of the coefficient matrix A for the following particular values of a: $a = -3$ for Case I, $a = 0$ for Case II, and $a = 3$ for Case III;

(c_2) sketch the phase portrait; indicate the direction of the "motion" on the integral lines; in Cases I and III, state which direction is "strongly" attracting/repelling and which direction is "weakly" attracting/repelling, and in your sketch indicate clearly to which straight line are the integral lines tangent at the origin.

(d) In parts (d_1)–(d_4) below, you will analyze in detail the case $a = a_1$.

(d_1) find the eigenvalues and eigenvectors of the coefficient matrix A;

Hint: In this case there is one attracting and one neutral direction.

(d_2) in the (x, y)-plane, indicate the position of the fixed points (i.e., the equilibrium solutions);

(d_3) solve explicitly the system (3) with an arbitrary initial condition, $\mathbf{x}(0) = \begin{pmatrix} x^{(0)} \\ y^{(0)} \end{pmatrix}$;

(d_4) from the solution found in part (d_3), observe that $x(t) + y(t) = x^{(0)} + y^{(0)}$ for any $t \in \mathbb{R}$, and use this fact to draw the complete phase portrait of the system.

Problem 3. [Attractivity vs. Lyapunov stability]
Consider the system

$$\begin{aligned} \dot{r} &= r(1 - r^2) \ , \\ \dot{\theta} &= 1 - \cos\theta \ , \end{aligned} \tag{4}$$

where (r, θ) represent the polar coordinates in $\mathbb{R}^2$. Sketch the phase portrait of the system (4) and show that the fixed point $(r^*, \theta^*) = (1, 0)$ is attracting, but not Lyapunov stable.

Problem 4. [Phase portrait of a conservative system]
Consider the second-order equation

$$\ddot{x} = x - x^2 \ . \tag{5}$$

(a) Rewrite the second-order equation (5) as a system of two first-order equations.

(b) Find and classify the fixed points of the system written in part (a). For each fixed point, find the eigenvalues and eigenvectors of the linearized system, and sketch the behavior of the solutions of the linearized system near the fixed point.

(c) Sketch the phase portrait of the system written in part (a).

(d) Find an expression of the potential energy $V(x)$ corresponding to (5).

(e) Find the equation for the homoclinic orbit that separates closed and non-closed trajectories in the phase space.

Problem 5. [Polar coordinates] **Only if you take the class as 5103!**

The polar coordinates (r, θ) are defined by $x = r\cos\theta$, $y = r\sin\theta$, and the inverse transform is given by

$$r = \sqrt{x^2 + y^2} \ ,$$

$$\theta = \begin{cases} \arctan \frac{y}{x} & \text{if } x > 0 \ , \\ \pi + \arctan \frac{y}{x} & \text{if } x < 0 \ , \\ \frac{\pi}{2} & \text{if } x = 0,\ y > 0 \ , \\ -\frac{\pi}{2} & \text{if } x = 0,\ y < 0 \ , \\ \text{undefined} & \text{if } x = 0,\ y = 0 \ . \end{cases}$$

Show that

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$$

in two different ways (by using the expressions for x and y in terms of r and θ, and by using the expressions for r and θ in terms of x and y).