MATH 4443 **Homework 3** **Due 2/8/18 (Thursday)**

Problem 1. Consider the sequence of functions

$$f_n(x) = \frac{nx}{1 + nx^2} \ .$$

(a) Find the pointwise limit of the sequence (f_n) for all $x \in [0, \infty)$.

(b) Is the convergence uniform on $(0, 1)$? Justify your answer.

(c) Is the convergence uniform on $(1, \infty)$? Prove your claim.

Problem 2. For each $n \in \mathbb{N}$ and $x \in [0, \infty)$, let

$$f_n(x) = \frac{x}{1 + x^n} \ .$$

(a) Find the pointwise limit of the sequence (f_n) on $[0, \infty)$.

(b) Explain how we know that the convergence *cannot* be uniform on $[0, \infty)$.

(c) Choose a smaller set over which the convergence is uniform and prove that this is indeed the case.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on all of $\mathbb{R}$, and define a sequence of functions by

$$f_n(x) = f\left(x + \tfrac{1}{n}\right) \ .$$

(a) Show that $f_n \to f$ uniformly.

(b) Give an example to show that this proposition fails if f is only assumed to be continuous and not uniformly continuous on $\mathbb{R}$.

Problem 4. Assume that (f_n) and (g_n) are uniformly convergent sequences of functions.

(a) Show that $(f_n + g_n)$ is a uniformly convergent sequence sequence of functions.

(b) Give an example to show that the product $(f_n g_n)$ may not converge uniformly.

(c) Prove that if there exists a constant M such that $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbb{N}$, then $(f_n g_n)$ converges uniformly.

Problem 5. We say that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies a *Hölder condition of order* α for some $\alpha > 0$ if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \le M\,|x - y|^{\alpha} \qquad \text{for all } x \text{ and } y \text{ in } \mathbb{R}\ . \tag{1}$$

(a) Prove that, if f satisfies Hölder condition of order $\alpha > 0$, then f is continuous on $\mathbb{R}$.

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$ and g' be bounded by some constant K (i.e., $|g'(z)| < K$ for all $z \in \mathbb{R}$).

Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$ be two arbitrary points, and assume (without loss of generality) that $x < y$. Apply the Mean Value Theorem to the function g on the interval $[x, y]$ to conclude that g satisfies Hölder condition of order $\alpha = 1$. What can you say about the value of the constant M (in the right-hand side of (1)) for the function g in this case?

(c) Show that the function $h(x) = 5|x|$ satisfies Hölder condition on $\mathbb{R}$. What are the values of the constants α and M (from the right-hand side of (1)) in this case?

(d) Show that, if f satisfies Hölder condition of order $\alpha > 1$, then f is differentiable. What can you say about the derivative of f?

Hint: Use the Hölder condition with $\alpha > 1$ to find the limit $\lim_{x\to y} \left| \frac{f(x) - f(y)}{x - y} \right|$, and explain what your result means.

Problem 6. Assume that $f_n \to f$ pointwise on a compact set K and assume that for each $x \in K$ the sequence $f_n(x)$ is increasing. Follow the steps below to show that if f_n and f are continuous on K, then the convergence is uniform.

(a) Set $g_n = f - f_n$ and translate the preceding hypothesis into statements about the sequence (g_n).

(b) Let $\varepsilon > 0$ be arbitrary, and define the sets

$$K_n := \{x \in K \ : \ g_n(x) \ge \varepsilon\}\ .$$

Argue that each set K_n is bounded.

(c) Argue that each set K_n is closed (e.g., assume that $(x_k)_{k\in\mathbb{N}}$ is a convergent sequence of numbers in K_n and show that $x := \lim_{k\to\infty} x_k \in K_n$).

(d) Argue that $K_1 \supseteq K_1 \supseteq K_2 \supseteq K_3 \cdots$, and use this observation to finish the argument.

Food for Thought: Abbott, Exercises 4.4.1, 4.4.6, 4.4.8, 6.2.12, 6.3.1, 6.3.4.