MATH 5163 Homework 3 Due Thu, 2/6/14

Problem 1. [Fourier transform applied to the advection-diffusion-decay equation] Let $u(x,t)$ be the concentration of a substance in an infinitely long narrow channel through which water flows in positive x-direction with constant speed c (called the *advection speed*). Since the channel is narrow, we can neglect the variation of the concentration within the cross-section of the channel, and think of u as a function only of the coordinate x along the channel and the time t. Assume also that the substance is decaying exponentially with time (which is exactly the case if it is radioactive or because of chemical reactions). This can be used as a rough model for the propagation of 4-methylcyclohexanemethanol that leaked into the Elk River on January 9, 2014 and contaminated the water in Charleston, West Virginia. The concentration u of the substance is described by the initial value problem

$$\begin{aligned} &u_t = \alpha^2 u_{xx} - c\,u_x - \gamma\,u\ , && x \in \mathbb{R}\ , \quad t > 0\ , \\ &u(x,0) = f(x)\ , && x \in \mathbb{R}\ , \end{aligned} \tag{1}$$

where α, v, and γ are positive constants, and $f(x)$ decays fast enough as $|x| \to \infty$ so that its Fourier transform $F(\xi)$ exists. The term $\alpha^2 u_{xx}$ in the right-hand side of the PDE corresponds to the diffusion of the substance in the water (which occurs whether or not the water is moving).

(a) Perform Fourier transform of all terms in the PDE in (1) to obtain an equation for the Fourier transform $U(\xi,t)$ of $u(x,t)$ which contains derivatives of U only with respect to t, but not with respect to ξ.

(b) Solve the equation for U obtained in part (a) and impose the condition on $U(\xi,0)$ coming from the boundary condition in (1) to obtain $U(\xi,t)$.

(c) Use that the Fourier transform of the Gaussian $\mathrm{e}^{-b^2x^2}$ (where $b > 0$) is equal to

$$\mathcal{F}\left(\mathrm{e}^{-b^2x^2}\right) = \frac{\sqrt{\pi}}{b}\,\mathrm{e}^{-\frac{\xi^2}{4b^2}}$$

in order to obtain the inverse Fourier transform $g(x,t)$ of the function

$$G(\xi,t) := \mathrm{e}^{-(\alpha^2\xi^2 + \mathrm{i}c\xi + \gamma)t}\ .$$

(d) Use the properties of convolution and your results from parts (b) and (c) to express the solution $u(x,t)$ of the IVP (1) as an integral.

(e) Compute explicitly the solution $u(x,t)$ of the IVP (1) if one unit of the contaminant was released instantaneously at time $t = 0$ at the point $x = 0$, so that $u(x,0) = \delta(x)$.

Problem 2. [Multidimensional Fourier Transform]

In this problem you will find the Fourier transform of the so-called *Yukawa potential*, named after the Japanese physicist Hideki Yukawa (1907–1981), recipient of the 1949 Nobel Prize for Physics for research on the theory of elementary particles. The Yukawa potential is important in plasma physics and elementary particle physics.

The Yukawa potential is given by the expression

$$u(\mathbf{r}) = \frac{\mathrm{e}^{-\alpha\rho}}{\rho} , \qquad \mathbf{r} \in \mathbb{R}^3 , \quad \rho := |\mathbf{r}| \tag{2}$$

(we omitted some inessential overall constants). Please follow the steps below.

(a) As a preliminary calculation, show that

$$\int_0^\pi \sin(\phi)\, \mathrm{e}^{-\mathrm{i}\xi\rho\cos\phi}\, \mathrm{d}\phi = \frac{2\sin(\xi\rho)}{\xi\rho}$$

(where $\mathrm{i} := \sqrt{-1}$). You will need to use the relation

$$\sin x = \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}} . \tag{3}$$

In the calculation of the integral, use what you know from Calculus, do not worry about the fact that some constants are not real numbers.

(b) Show that

$$\int_0^\infty \mathrm{e}^{-\alpha\rho} \sin(\xi\rho)\, \mathrm{d}\rho = \frac{\xi}{\xi^2 + \alpha^2} .$$

Hint: You can either use integration by parts (in a little bit tricky way) or the following method: write $\sin(\xi\rho)$ in terms of $\mathrm{e}^{\pm\mathrm{i}\xi\rho}$ as in (3), then the integral becomes a sum of two easy integrals of exponents.

(c) In the triple Fourier transform $U(\boldsymbol{\xi})$ of $u(\mathbf{r})$, change the Cartesian coordinates to spherical coordinates, (ρ, θ, ϕ), by

$$\begin{aligned} x &= \rho \sin\phi \cos\theta , \\ y &= \rho \sin\phi \sin\theta , \\ z &= \rho \cos\phi . \end{aligned}$$

Show that if you take $\boldsymbol{\xi}$ to be in the direction of the positive z-axis (which you can do without loss of generality), then

$$\begin{aligned} U(\boldsymbol{\xi}) &= \iiint_{\mathbb{R}^3} u(\mathbf{r})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{r}}\, \mathrm{d}^3\mathbf{r} \\ &= \frac{4\pi}{\xi^2 + \alpha^2} , \end{aligned}$$

where $\xi := |\boldsymbol{\xi}|$. Please write clearly all details of your calculations.

Problem 3. [Exponential decay of the norm of a solution of the heat equation] Let a and α be positive numbers. In this problem you will study the decay of the L^2-norm of the solution of the following initial-boundary value problem (IBVP) for the heat equation on the domain $(x,t) \in (0,a) \times (0,\infty)$ with homogeneous (i.e., zero) Dirichlet boundary conditions:

$$\begin{aligned} &u_t = \alpha^2 u_{xx} \ , && x \in (0,a) \ , \quad t > 0 \ , \\ &u(x,0) = \frac{1}{a^2}\, x(a-x) \ , && x \in [0,a] \ , \\ &u(0,t) = 0 \ , \quad u(a,t) = 0 \ , && t > 0 \ . \end{aligned} \tag{4}$$

The physical interpretation of the IBVP (4) is the following: $u(x,t)$ is the temperature in a thin rod with thermally insulated side walls, with initial temperature $u(x,0) = \frac{1}{a^2}\, x(a-x)$, and with the temperature at the ends kept at zero. Physically, it is clear that the temperature in the rod will tend to zero. Below you will use some inequalities to obtain an upper bound on the L^2-norm of the solution of the IBVP (4) without knowing the solution $u(x,t)$ explicitly. Assume that the solution $u(x,t)$ is twice continuously differentiable with respect to x and continuously differentiable with respect to t. Please follow the steps below. For an arbitrary function v of the spatial coordinate x and the time t, we use the notations

$$\|v(\cdot,t)\|_{L^2([0,a])} := \left(\int_0^a |v(x,t)|^2 \, \mathrm{d}x \right)^{1/2} \ , \qquad \|v(\cdot,t)\|_{C^0([0,a])} := \max_{x\in[0,a]} |v(\cdot,t)| \tag{5}$$

to denote the corresponding norms over the spatial variable.

(a) Make sure that you know how to prove that

$$\|u(\cdot,0)\|_{L^2([0,a])} = \sqrt{\frac{a}{30}} \ .$$

There is no need to attach your calculations to the homework!

(b) Use the PDE and the boundary conditions to prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(\cdot,t)\|^2_{L^2([0,a])} \right) = -2\alpha^2 \, \|u_x(\cdot,t)\|^2_{L^2([0,a])} \ . \tag{6}$$

Hint: You will have to use the relation (easily obtained by integration by parts)

$$\int_0^a u(x,t)\, u_{xx}(x,t)\, \mathrm{d}x = \Big[u(x,t)\, u_x(x,t) \Big]\Big|_{x=0}^{a} - \int_0^a |u_x(x,t)|^2 \, \mathrm{d}x \ .$$

Note that this equality is the one-dimensional version of the Green's formula

$$\int_U \left(|\nabla u|^2 + u\, \Delta u \right) \mathrm{d}V = \oint_{\partial U} u \, \frac{\partial u}{\partial \nu} \, \mathrm{d}S \quad \left(= \oint_{\partial U} u \, \nabla u \cdot \mathrm{d}\mathbf{S} \right) \ .$$

(c) Explain what was used in steps (I) and (III) below:

$$
\begin{aligned}
|u(x,t)| &\overset{\text{(I)}}{=} \left| \int_0^x u_x(y,t)\,\mathrm{d}y \right| = \left| \int_0^x 1 \cdot u_x(y,t)\,\mathrm{d}y \right| \\
&\overset{\text{(II)}}{\le} \left(\int_0^x 1^2\,\mathrm{d}y \right)^{1/2} \left(\int_0^x |u_x(y,t)|^2\,\mathrm{d}y \right)^{1/2} \\
&\overset{\text{(III)}}{\le} \sqrt{x}\, \|u_x(\cdot,t)\|_{L^2([0,a])}
\end{aligned}
$$

(step (II) is the Cauchy-Schwarz inequality). How does this implies the inequality

$$
\|u(\cdot,t)\|_{C^0([0,a])} \le \sqrt{a}\, \|u_x(\cdot,t)\|_{L^2([0,a])} \ ? \tag{7}
$$

(d) Write $\|u(\cdot,t)\|^2_{L^2([0,a])}$ explicitly as an integral (cf. (5)) and use the bound (7) in an obvious way to show that

$$
\|u(\cdot,t)\|^2_{L^2([0,a])} \le a^2\, \|u_x(\cdot,t)\|^2_{L^2([0,a])} \ .
$$

(e) Combine the results of (b) and (d) to prove the bound

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u(\cdot,t)\|^2_{L^2([0,a])} \Big) \le -\frac{2\alpha^2}{a^2}\, \|u(\cdot,t)\|^2_{L^2([0,a])} \ . \tag{8}
$$

(f) One form of the useful *Gronwall's inequality* states that if Ψ is a C^1 function of one variable satisfying the differential inequality

$$
\Psi'(t) \le \beta(t)\,\Psi(t) \ ,
$$

then the following bound holds for any $t > 0$:

$$
\Psi(t) \le \Psi(0)\, \exp\left(\int_0^t \beta(s)\,\mathrm{d}s \right)
$$

(cf. Lemma 9.3 in Salsa's book). Apply this to (8) and use the result from part (a) to show that the L^2-norm over x of the solution $u(x,t)$ of the IBVP (4) decays with time as

$$
\|u(\cdot,t)\|_{L^2([0,a])} \le \sqrt{\frac{a}{30}}\, \exp\left(-\frac{\alpha^2}{a^2}\, t \right) \ .
$$