MATH 5403 Homework 3 Due Tue, 10/29/19

Problem 1. [Lagrangian and Hamiltonian formalisms]
In this problem you will consider the Lagrangian and the Hamiltonian description of the flight of a stone (a point particle) that at time 0 is thrown from the ground level with some initial velocity (with positive vertical component) and we want that at time T it lands at a point that is at a distance ℓ from the initial point. Let us choose a coordinate system with the y_1-axis horizontal, the y_2-axis vertical, and let the position of the stone at time t be $\mathbf{y}(t) = (y_1(t), y_2(t))$. Let the initial point have coordinates $\mathbf{y}(0) = \mathbf{y}_0 = (y_{10}, y_{20}) = (0, 0)$, while the final point have coordinates $\mathbf{y}(T) = (\ell, 0)$, as shown in Figure 1.

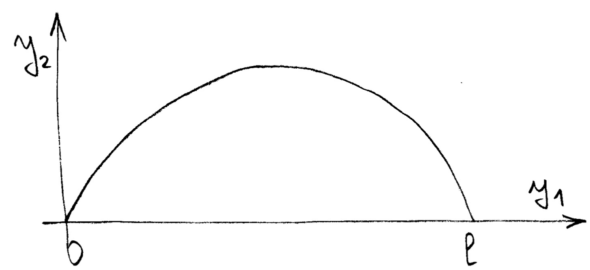

Figure 1: Coordinate system used in the problem.

The action for this system is $J[\mathbf{y}] = \displaystyle\int_0^T L(t, \mathbf{y}, \mathbf{y}')\, \mathrm{d}t$, where

$$L(t, \mathbf{y}, \mathbf{y}') = L(t, y_1, y_2, y_1', y_2') = \frac{m}{2}\left[(y_1')^2 + (y_2')^2\right] - mgy_2 \tag{1}$$

is the Lagrangian, which is equal to the difference between the kinetic and the potential energy of the stone.

(a) Write down the Euler-Lagrange equations for the Lagrangian (1), find their general solutions and their particular solution satisfying the boundary conditions $\mathbf{y}(0) = (0, 0)$, $\mathbf{y}(T) = (\ell, 0)$.

(b) What kind of curve is the trajectory of the stone that you obtained in part (a)? Justify your answer.

(c) Find the generalized momenta, $\mathbf{p} = \dfrac{\partial L}{\partial \mathbf{y}'}$, i.e., $(p_1, p_2) = \left(\dfrac{\partial L}{\partial y_1'}, \dfrac{\partial L}{\partial y_2'}\right)$, and write down the Hamiltonian $H(t, \mathbf{y}, \mathbf{p})$ of the system.

(d) Write down the Hamilton's equations, $\dfrac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \dfrac{\partial H}{\partial \mathbf{p}}$, $\dfrac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\dfrac{\partial H}{\partial \mathbf{y}}$, and find their general solution.

(e) Impose the boundary conditions from part (a) to find the solution $(\mathbf{y}(t), \mathbf{p}(t))$ of the Hamilton's equations that satisfies them.

Problem 2. [Flow in the phase space; Poisson brackets]

This problem is a continuation of Problem 1, where you found the Hamiltonian $H(t, \mathbf{y}, \mathbf{p})$ of the physical system and wrote down the corresponding Hamilton's equations.

(a) The solution $(\mathbf{y}(t), \mathbf{p}(t))$ of the Hamilton's equations that you found in part (d) of Problem 1 can be written as

$$\begin{aligned} y_1(t) &= \frac{p_{10}}{m}\, t + y_{10} \ , \\ y_2(t) &= -\frac{g}{2}\, t^2 + \frac{p_{20}}{m}\, t + y_{20} \ , \\ p_1(t) &= p_{10} \ , \\ p_2(t) &= -mgt + p_{20} \ . \end{aligned} \tag{2}$$

where $(\mathbf{y}_0, \mathbf{p}_0) = (y_{10}, y_{20}, p_{10}, p_{20})$ are the initial values of $(\mathbf{y}(t), \mathbf{p}(t))$ (at time 0). Let $\mathbf{\Phi}_t : \mathbb{R}^4 \to \mathbb{R}^4$ be the flow of the Hamiltonian system, i.e., $\mathbf{\Phi}_t(\mathbf{y}_0, \mathbf{p}_0) = (\mathbf{y}(t), \mathbf{p}(t))$, where $(\mathbf{y}(t), \mathbf{p}(t))$ are given by (2). Prove that the flow $\mathbf{\Phi}_t$ satisfies the semigroup property $\mathbf{\Phi}_s \circ \mathbf{\Phi}_t = \mathbf{\Phi}_{s+t}$.

(b) Consider the observable $G : \mathbb{R}^4 \to \mathbb{R}$ defined as $G(\mathbf{y}, \mathbf{p}) = y_1 p_2 - p_1 y_2$. Directly from the expression for the rate of change of an observable with time in terms of Poisson brackets (without using the explicit expression (2) for the flow $\mathbf{\Phi}_t$), express the time rate of change of the observable G, i.e., $\dfrac{\mathrm{d}}{\mathrm{d}t}\,(G \circ \mathbf{\Phi}_t)$.

(c) Now use the explicit expression (2) for the flow $\mathbf{\Phi}_t$ to find $G \circ \mathbf{\Phi}_t(\mathbf{y}_0, \mathbf{p}_0)$ as an explicit function of time. Differentiate this expression to find the rate of change of G with time.

(d) In parts (b) and (c) of this problem you are computing the same quantity. Prove that your answers agree.

Problem 3. [Legendre transform]

Let the function $f : (0, 4) \to \mathbb{R}$ be given by

$$f(\xi) = \begin{cases} \frac{1}{4}\xi^2 \ , & \xi \in (0, 2) \ , \\ \xi - 1 \ , & \xi \in [2, 3] \ , \\ \xi^2 - 5\xi + 8 \ , & \xi \in (3, 4) \ . \end{cases} \tag{3}$$

This function is C^1 and convex (this is quite obvious, so you do not have to prove it); the graph of the function is given in Figure 2. (The gaps in the graph are a problem in Mathematica that I could not fix.)

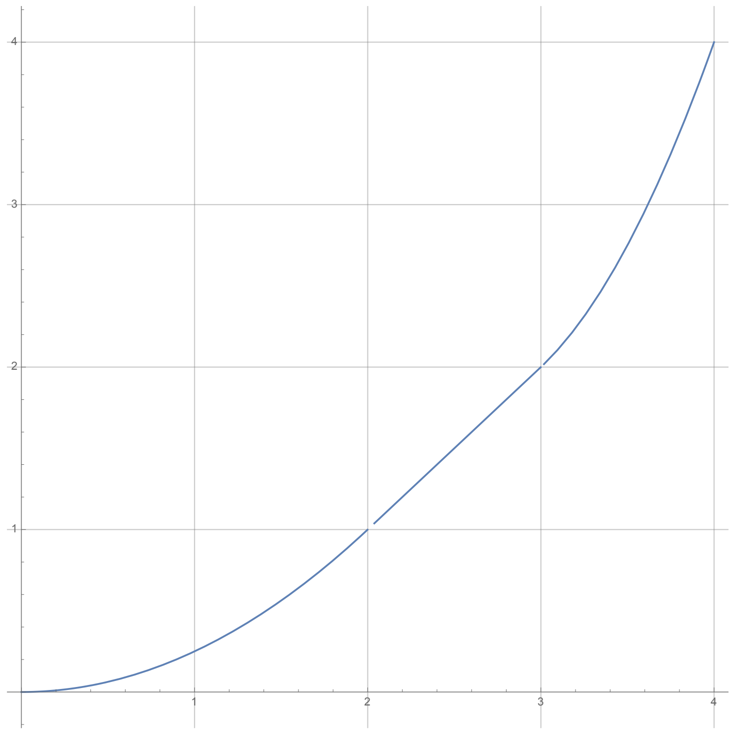

Figure 2: Graph of the function $f(\xi)$ defined in (3).

(a) Compute the Legendre transform, $H(p)$, of the function $f(\xi)$ from (3). Since the slope of the tangent to the graph of $f(\xi)$ is in the interval $(0, 3)$ (as you can easily check), think of $H(p)$ as a function $H : (0, 3) \to \mathbb{R}$.

Hint: Do the calculation separately for $p \in (0, 1)$ and for $p \in (1, 3)$. Pay special attention to the value $p = 1$; explain how you obtained $H(1)$. The expression for the Legendre transform in the interval $p \in (1, 3)$ is $H(p) = \frac{1}{4}(p^2 + 10p - 7)$ (I want to see your calculations).

(b) The graph of $H(p)$ is drawn in Figure 3. Notice that $H(p)$ is continuous, but not C^1 because of the sudden change of the slope at $p = 1$; it is also a convex function. Perform Legendre transform of the function $H(p)$. Please show me your calculations in detail.

(c) Does your result in part (b) surprise you? Why or why not?

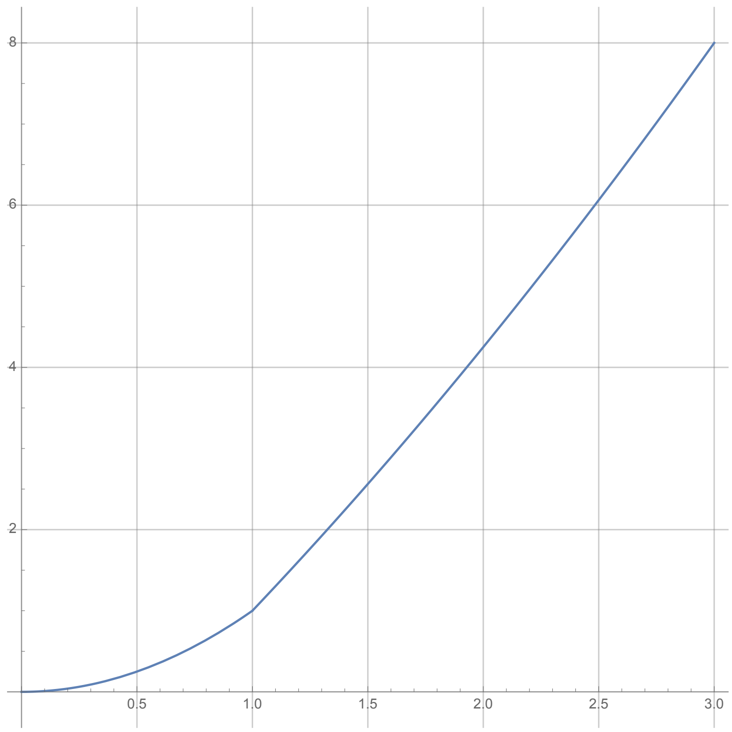

Figure 3: Graph of the Legendre transform $H(p)$ of the function $f(\xi)$ from (3).