

**Problem 1.** Find the disk of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n! z^n}{n^n} .$$

*Hint:* Recall the fact that  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$ .

**Problem 2.**

- (a) Find the Taylor expansion of the function  $f(z) = \frac{1}{1+z^2}$  about  $z = 0$  by using the formula for the sum of a geometric series applied to  $\frac{1}{1+z^2} = \frac{1}{1-(-z^2)}$ .
- (b) Use your result from (a) and the fact that  $\arctan w = \int_0^w \frac{dz}{1+z^2}$  to obtain the Taylor expansion of  $\arctan w$  about  $w = 0$ .

*Remark:* It can be proven that one can integrate a convergent power series term by term, and the new series has the same disk of convergence.

**Problem 3.** Consider the function

$$f(z) = \frac{e^{3z}}{(z-5)^4} .$$

- (a) What is the nature of the singularity of  $f(z)$  at  $a = 5$ ?  
*Hint:* It is easy to answer this question by using the fact stated on page 914 of the book.
- (b) Find the Laurent expansion of  $f(z)$  about  $a = 5$ .  
*Hint:* Look at Example 5 on page 908 of the book.
- (c) Find the residue of  $f(z)$  at  $a = 5$  from your result in part (b).
- (d) Compute the residue of  $f(z)$  at  $a = 5$  by using the formula

$$\text{Res } f(a) = \frac{1}{(N-1)!} \lim_{z \rightarrow a} \frac{d^{N-1}}{dz^{N-1}} [(z-a)^N f(z)] .$$

- (e) Use the Residue Theorem to compute  $\oint_C \frac{e^{3z}}{(z-5)^4} dz$  where  $C$  is the contour  $|z - 3 - i| = 4$ , traversed in positive direction. Draw the contour in  $\mathbb{C}$ .
- (f) Compute the value of the integral from part (e) by using the generalized Cauchy integral formula (equation (7) on page 897 of the book).

**Problem 4.** Apply the Residue Theorem to evaluate

$$\oint_C \frac{1 - \cos z}{z^4} dz ,$$

where  $C$  is the ellipse  $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$ , traversed in positive direction.

*Hint:* First check carefully the order of the pole at 0.

**Problem 5.**

(a) Show that  $\int_0^{2\pi} \frac{d\theta}{3 + \cos \theta} = \frac{\pi}{\sqrt{2}}$ .

(b) Compute the value of the integral  $\int_0^\pi \frac{d\theta}{3 + \cos \theta} = \frac{\pi}{\sqrt{2}}$ .

*Hint:* Your result from part (a) will be useful; please write your reasoning in detail.

**Problem 6.** Show that, if  $f$  is analytic on and inside a simple closed contour  $C$ , and if the point  $a$  is not on the contour  $C$ , then

$$\oint_C \frac{f'(z)}{z - a} dz = \oint_C \frac{f(z)}{(z - a)^2} dz .$$

*Hint:* Apply the generalization of the Cauchy Integral Formula.

*Remark:* Consider separately the case when  $a$  is outside of the contour  $C$ , and when  $a$  is inside  $C$ . Also, recall that the derivative of an analytic function is also analytic.