MATH 3423 **Homework 4** **Due Fri, 09/30/16**

Problem 1. Find the disk of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n!\, z^n}{n^n} \ .$$

Hint: Recall the fact that $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \mathrm{e}$.

Problem 2.

(a) Find the Taylor expansion of the function $f(z) = \frac{1}{1+z^2}$ about $z = 0$ by using the formula for the sum of a geometric series applied to $\frac{1}{1+z^2} = \frac{1}{1-(-z^2)}$.

(b) Use your result from (a) and the fact that $\arctan w = \int_0^w \frac{\mathrm{d}z}{1+z^2}$ to obtain the Taylor expansion of $\arctan w$ about $w = 0$.

Remark: It can be proven that one can integrate a convergent power series term by term, and the new series has the same disk of convergence.

Problem 3. Consider the function

$$f(z) = \frac{\mathrm{e}^{3z}}{(z-5)^4} \ .$$

(a) What is the nature of the singularity of $f(z)$ at $a = 5$?
Hint: It is easy to answer this question by using the fact stated on page 914 of the book.

(b) Find the Laurent expansion of $f(z)$ about $a = 5$.
Hint: Look at Example 5 on page 908 of the book.

(c) Find the residue of $f(z)$ at $a = 5$ from your result in part (c).

(d) Compute the residue of $f(z)$ at $a = 5$ by using the formula

$$\operatorname{Res} f(a) = \frac{1}{(N-1)!} \lim_{z\to a} \frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}} \left[(z-a)^N f(z)\right] \ .$$

(e) Use the Residue Theorem to compute $\oint_C \frac{\mathrm{e}^{3z}}{(z-5)^4}\, \mathrm{d}z$ where C is the contour $|z-3-i| = 4$, traversed in positive direction. Draw the contour in $\mathbb{C}$.

(f) Compute the value of the integral from part (e) by using the generalized Cauchy integral formula (equation (7) on page 897 of the book).

Problem 4. Apply the Residue Theorem to evaluate

$$\oint_C \frac{1-\cos z}{z^4}\,\mathrm{d}z \ ,$$

where C is the ellipse $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$, traversed in positive direction.

Hint: First check carefully the order of the pole at 0.

Problem 5.

(a) Show that $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{3+\cos\theta} = \frac{\pi}{\sqrt{2}}$.

(b) Compute the value of the integral $\displaystyle\int_0^{\pi} \frac{\mathrm{d}\theta}{3+\cos\theta} = \frac{\pi}{\sqrt{2}}$.

Hint: Your result from part (a) will be useful; please write your reasoning in detail.

Problem 6. Show that, if f is analytic on and inside a simple closed contour C, and if the point a is not on the contour C, then

$$\oint_C \frac{f'(z)}{z-a}\,\mathrm{d}z = \oint_C \frac{f(z)}{(z-a)^2}\,\mathrm{d}z \ .$$

Hint: Apply the generalization of the Cauchy Integral Formula.

Remark: Consider separately the case when a is outside of the contour C, a and when a is inside C. Also, recall that the derivative of an analytic function is also analytic.