

**Problem 1.** Assume that the sequence  $\{p_n\}_{n=0}^{\infty}$  is generated by some iterative method for finding a root of an equation. Also assume that we know that the sequence  $\{p_n\}_{n=0}^{\infty}$  converges to some number  $p$  of some order  $\alpha$  with some asymptotic error constant  $\lambda$ , but we don't know the values of  $\alpha$  and  $\lambda$ . The goal of this problem is to develop a method for determining the numerical value of  $\alpha$  from the numerical values of the members of the sequence  $\{p_n\}_{n=0}^{\infty}$ .

Let  $E_n := |p_n - p|$  be the error at the  $n$ th step of the iteration, and define  $\ell_n := \ln E_n$ .

- (a) Show that for large  $n$ , the following approximate identity holds:

$$\ell_n - \alpha \ell_{n-1} \approx \ln \lambda .$$

*Hint:* Just look at the definition of order of convergence.

- (b) Using the approximate identity derived in (a) show that

$$\alpha \approx \frac{\ell_n - \ell_{n+1}}{\ell_{n-1} - \ell_n} .$$

Note that this approximate formula for  $\alpha$  does not depend on the base of the logarithms; if  $\ell_n$  is defined as the log base 10 of  $E_n$ , the formula will remain the same.

- (c) The Mathematica notebook `rate_of_convergence1.nb` (which you can download from the class website) computes the value of the solution of the equation  $\sin x + x = 1$  using (i) Newton's method, (ii) the secant method, (iii) the fixed point iteration method, (iv) the fixed point iteration method combined with Aitken extrapolation, and (v) the fixed point iteration method combined with Steffensen's method for accelerating convergence. Run all the programs from the notebook and use the formula for  $\alpha$  derived in part (b) to find empirically the numerical values of  $\alpha$  for all these methods. Does the values of  $\alpha$  you have obtained match the theoretical predictions? Discuss briefly. One can prove (but you do *not* need to do this!) that the order of convergence of the secant method is

$$\alpha_{\text{secant}}^{(\text{theor})} = \frac{1}{2}(\sqrt{5} + 1) \approx 1.61803398874 \dots .$$

*Note:* Since we are using a very high accuracy (namely, 10000 digits), you have to be patient when running the programs; make sure that a program is finished before you run the next one. There is no need to attach the printout of running the programs – it will be too long. Give only the numbers that you use and show your calculations in detail. Recall that to perform some calculation in Mathematica, you need to put the cursor on that part of the program, press SHIFT, and while holding it down, press RETURN. Finally, note that in the notebook `rate_of_convergence1.nb` logarithms base 10 are used.

**Problem 2.** In this problem you will develop a modification the Newton's method to find a zero of multiplicity  $m \geq 1$  of the equation  $f(x) = 0$ .

- (a) Let the sequence  $\{p_n\}_{n \in \mathbb{N}}$  be defined by the recurrence relation  $p_n = g(p_{n-1})$  (and  $p_0$  is some initial guess), where  $g(x) = x - \frac{m f(x)}{f'(x)}$ . Prove that, if the point  $p$  is a zero of multiplicity  $m$  of  $f$ , then  $g'(p) = 0$ . This implies that the sequence  $\{p_n\}_{n \in \mathbb{N}}$  converges faster than linearly.

*Hint:* The calculations you need to do are very similar to the ones on pages 80–81 of the book involving the function  $\mu(x)$ .

- (b) Find the multiplicity of the root  $\frac{\pi}{2}$  of the equation  $(x - \frac{\pi}{2})^2 \cos x = 0$ .
- (c) The following Mathematica code similar to the codes from Problem 1

```
p = N[3, 50000];
m = (**put here your answer from part (b)**) ;
f[x_] := (x - Pi/2)^2 * Cos[x];
For[i = 1, i <= 11, i++,
  { p = p - m*f[p]/f'[p],
    error = Abs[p - Pi/2],
    Print[i], Print[Log[1.0*error]]
  }
]
```

can be used (after an appropriate modification) to find empirically the order of convergence of the method. What is the order of the convergence that you observe? Explain briefly your reasoning. (Again, be patient – Mathematica can take a while because you are doing the calculations with accuracy or 50000 decimal digits!)

- (d) The number  $\frac{\pi}{2}$  is a root of the equation  $(x - \frac{\pi}{2})^4 \cos x = 0$  of multiplicity 5. Run the Mathematica code from part (c) after modifying it appropriately and use the output to find the order of convergence in this case. What do you observe?
- (e) Modify the Matlab code `newton.m` to write a code `newton_m.m` that implements the algorithm from part (a) for finding zeros of multiplicity  $m$  of the equation  $f(x) = 0$ . The first line of your code must be

```
function r = newton_m( fun, funder, m, xinit, tol, nmax, verbose)
```

Note that in Matlab the name of a file must be the same as the name of the code in it.

- (f) One can estimate roughly the order of convergence  $m$  of an iterative method as follows. If after iterating enough number of times the error is significantly smaller than 1 (say, 0.01 or smaller), then the number of correct digits of  $p_{n+1}$  is roughly  $m$  times the number of correct digits of  $p_n$ . You can observe this, in the following quadratically convergent sequence (coming from using Newton's method to find  $\sqrt{10}$  as in Problem 4 of Homework 3)

