MATH 4073 Homework 4 Due Wed, 9/18/13

Problem 1. Directly from the definition, find the rates of convergence α and the asymptotic error constants λ for each of the sequences (all of which tend to 0)

$$\text{(a)}\ \ x_n = \frac{1}{n^2}\ ; \qquad \text{(b)}\ \ x_n = 7^{-n}\ ; \qquad \text{(c)}\ \ x_n = 10^{-5^n}\ .$$

Problem 2. In this problem we will find the value of the number $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}}}$, and will study how a certain sequence converges to it.

(a) Show that the numbers 2 and 4 satisfy the equation $\sqrt{2}^{\,x} = x$.

(b) Consider the function $g(x) := \sqrt{2}^{\,x}$, $x \geq 0$. Use derivatives to show that g is an increasing concave up function for $x \in [0, \infty)$. (To differentiate g, note that $\sqrt{2}^{\,x} = \mathrm{e}^{\frac{\ln 2}{2}x}$.)

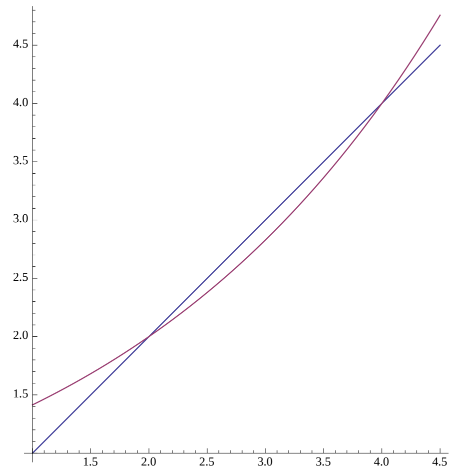

Figure 1: Plots of the diagonal and the graph of $g(x) = \sqrt{2}^{\,x}$.

(c) Use what you found in (b) to convince me that the equation $\sqrt{2}^{\,x} = x$ has no other solutions in $[0, \infty)$ except 2 and 4.

(d) Is the fixed point $x = 2$ of the function attracting or repelling? How about $x = 4$? Justify your claims (a pictorial "proof" is enough).

(e) Let $x_0 := 1$, $x_n := \sqrt{2}^{\,x_{n-1}}$ for $n \geq 1$. Perform several iterations by using Mathematica, MATLAB, or any other software. Does the sequence $(x_n)_{n=0}^{\infty}$ seem to converge? To what value? (There is no need to attach a printout, just tell me what you observe.)

(f) Let x_* be the limit of the sequence defined in (e). Show theoretically that the limit

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_*|}{|x_n - x_*|}$$

exists. What does the theory predict about the value of this limit?

Problem 3. In this problem you will construct a piecewise-linear and a quadratic Lagrange interpolating polynomials for the function $\cos(\pi x)$ on the interval $[0, \frac{1}{2}]$.

Let $f(x) = \cos(\pi x)$, and $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{2}$. You will need to use the values of $f(x)$ at these points: $y_0 = \cos(0) = 1$, $y_1 = \cos\frac{\pi}{3} = \frac{1}{2}$ and $y_2 = \cos\frac{\pi}{2} = 0$. Figure 2 shows the graphs of $f(x)$ and the interpolating polynomials you will compute.

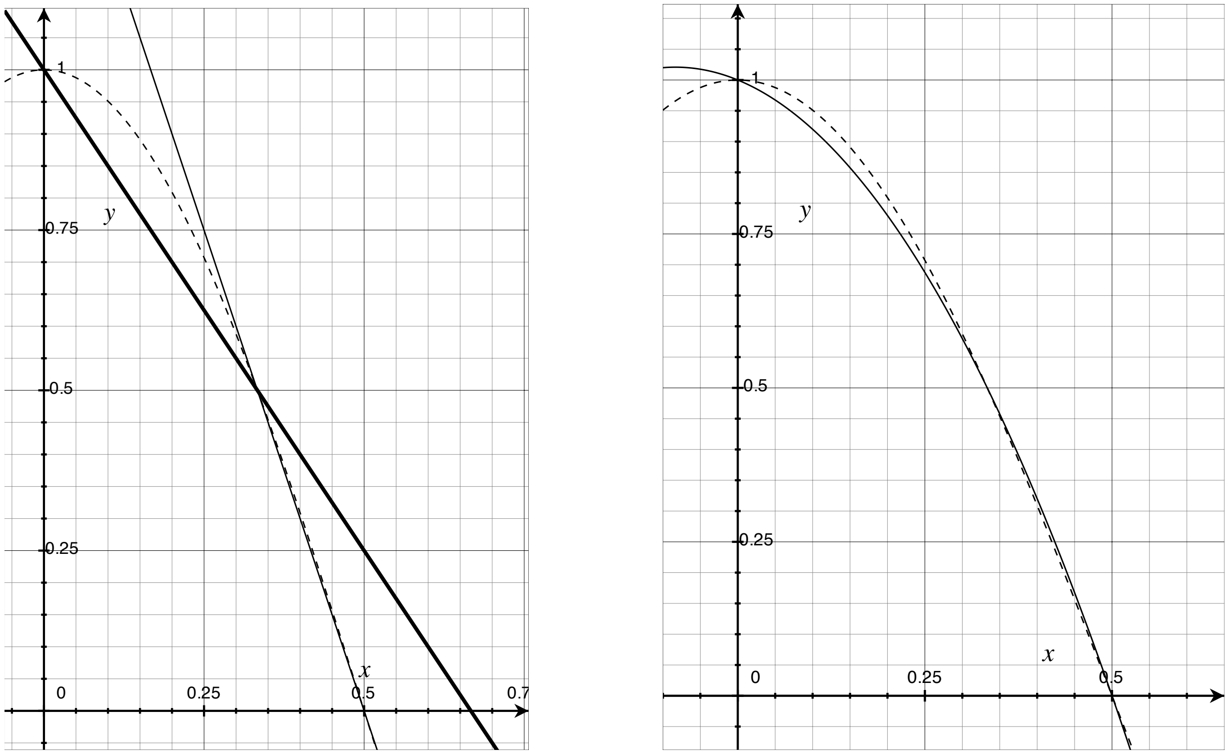

Figure 2: (LEFT) Piecewise-linear interpolation of $\cos\pi x$ (the function $\cos\pi x$ is plotted with dashed line, the interpolating linear function on $[0, \frac{1}{3}]$ with a thick solid line, and the interpolating linear function on $[\frac{1}{3}, \frac{1}{2}]$ with a thin solid line). (RIGHT) Quadratic interpolation of $\cos\pi x$ on $[0, \frac{1}{2}]$ using the points $(0, 1)$, $(\frac{1}{3}, \frac{1}{2})$, and $(\frac{1}{2}, 0)$.

(A) Piecewise-linear interpolation.

(A1) Find the piecewise-linear interpolation of the function $f(x)$ on the interval $[0, \frac{1}{2}]$ based on the values of the function at the points 0, $\frac{1}{3}$, and $\frac{1}{2}$. Naturally, you will have to find two different first-order Lagrange polynomials – one in the interval $[0, \frac{1}{3}]$, and another one on $[\frac{1}{3}, \frac{1}{2}]$:

$$P_{\text{piece-lin}}(x) = \begin{cases} b_1 x + c_1 \ , & x \in [0, \frac{1}{3}] \ , \\ b_2 x + c_2 \ , & x \in [\frac{1}{3}, \frac{1}{2}] \ . \end{cases}$$

(A2) Use the piecewise-linear interpolating polynomial found in part (A1) to compute the approximate value of $\cos\frac{\pi}{6}$, and use it to find the numerical value of the absolute error $|\cos\frac{\pi}{6} - P_{\text{piece-lin}}(\frac{1}{6})|$.

(B) Quadratic Lagrange interpolation.

(B1) Construct the Lagrange interpolating polynomial of degree at most 2, $P_2(x)$, to $\cos(\pi x)$ in the interval $[0, \frac{1}{2}]$. The polynomial $P_2(x)$ is the only quadratic function whose graph goes through the points $(0, \cos 0) = (0, 1)$, $(\frac{1}{3}, \cos\frac{\pi}{3}) = (\frac{1}{3}, \frac{1}{2})$ and $(\frac{1}{2}, \cos\frac{\pi}{2}) = (\frac{1}{2}, 0)$.

(B2) Use the quadratic Lagrange interpolating polynomial found in part (B1) to compute the approximate value of $\cos\frac{\pi}{6}$. Find the numerical value of the absolute error $|\cos\frac{\pi}{6} - P_2(\frac{1}{6})|$.

Problem 4. This problem is a continuation of Problem 3. Let

$$I_{\text{exact}} := \int_0^{\frac{1}{2}} \cos(\pi x)\, dx \ ,$$

$$I_{\text{piece-lin}} := \int_0^{\frac{1}{2}} P_{\text{piece-lin}}(x)\, dx \ , \qquad I_{\text{quadr}} := \int_0^{\frac{1}{2}} P_2(x)\, dx$$

be the definite integrals from 0 to $\frac{1}{2}$ of the function $\cos(\pi x)$ and the piecewise-linear and the quadratic interpolating functions you computed in Problem 3.

(a) Without computing anything, decide which of the numbers I_{exact} and $I_{\text{piece-lin}}$ is larger.

Why can't you use the similar simple reasoning to decide which of the numbers I_{exact} and I_{quadr} is larger?

Hint: Look at Figure 2 and think about the concavity of $\cos(\pi x)$ for $x \in [0, \frac{1}{2}]$.

(b) Compute the numerical values of I_{exact}, $I_{\text{piece-lin}}$, and I_{quadr}. Was your prediction in part (a) correct?

Hint: Be careful when you compute $I_{\text{piece-lin}}$; the value of this integral is $\frac{7}{24} \approx 0.291667$.

(c) Compute the numerical values of the absolute errors in approximating I_{exact} by $I_{\text{piece-lin}}$ and by I_{quadr}.

Problem 5. The purpose of this problem is to construct and study the Newton's divided difference form of the interpolating polynomial,

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, \ldots, x_k] \prod_{j=0}^{k-1} (x - x_j) \ ,$$

to the function $f(x) = \cos(\pi x)$. The points x_i, $i = 0, 1, 2, 3$ used to construct the interpolating polynomial are given in the table below (x_0, x_1 and x_2 are the same as in the previous problems). The graphs of the function $f(x)$ and the interpolating polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$, and $P_3(x)$ are plotted in Figure 3 below.

(a) Compute the missing entries in the divided differences table below. Write your calculations clearly and leave the coefficients in symbolic form (i.e., do not compute the *numerical values* of things like $12(8\sqrt{2} - 11)$).

x_i	0^{th} order	1^{st} order	2^{nd} order	3^{rd} order
$x_0 = 0$	$f[x_0] = 1$			
		$f[x_0, x_1] = ?$		
$x_1 = \frac{1}{3}$	$f[x_1] = \frac{1}{2}$		$f[x_0, x_1, x_2] = ?$	
		$f[x_1, x_2] = ?$		$f[x_0, x_1, x_2, x_3] = 12(8\sqrt{2} - 11)$
$x_2 = \frac{1}{2}$	$f[x_2] = 0$		$f[x_1, x_2, x_3] = 12(2\sqrt{2} - 3)$	
		$f[x_2, x_3] = -2\sqrt{2}$		
$x_3 = \frac{1}{4}$	$f[x_3] = ?$			

(b) Write down the interpolating polynomial $P_0(x)$ based on the values in the divided differences table above.

(c) Similarly to part (b), write down the interpolating polynomial $P_1(x)$ based on the values in the divided differences table above.

(d) Similarly to part (b), write down the interpolating polynomial $P_2(x)$ based on the values in the divided differences table above. Do *not* expand it – just substitute the coefficients in the Newton's divided difference interpolating polynomial with the corresponding entries from the table.

(e) Similarly to part (b), write down the interpolating polynomial $P_3(x)$ based on the values in the divided differences table above. Do *not* expand the polynomial!

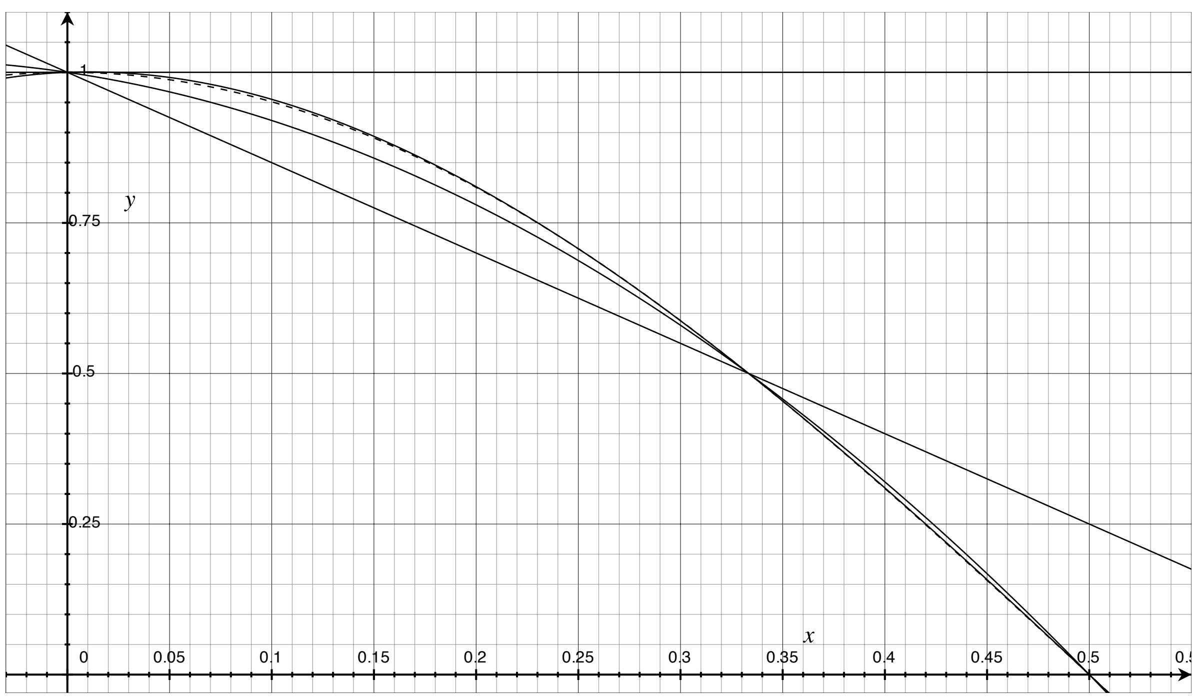

Figure 3: Graphs of the function $f(x)$ (the dashed line) and the interpolating polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$, and $P_3(x)$.