MATH 4093/5093 **Homework 4** **Due Fri, 2/25/2011**

Problem 1. Directly from the definition, find the rates of convergence α and the asymptotic error constants λ for each of the sequences (all of which tend to 0)

$$\text{(a)}\ \ p_n = \frac{1}{n^2}\ ; \qquad \text{(b)}\ \ p_n = 7^{-n}\ ; \qquad \text{(c)}\ \ p_n = 10^{-5^n}\ .$$

Problem 2. The concept of rate of convergence (in particular, the order of convergence α and the asymptotic error constant λ) are very important when one is using an *iterative method*, i.e., a method in which the exact solution of the problem is found as a limit of a sequence of approximate values. If the exact value p is a limit of a sequence $\{p_n\}_{n=0}^{\infty}$ of approximate values, then the *error* at the nth step of the iteration is $E_n := |p_n - p|$. The rate of decreasing of the error is one of the most important characteristics of iterative methods of numerical computations.

Assume that the sequence $\{p_n\}_{n=0}^{\infty}$ is generated by some iterative method for finding a root of an equation. Also assume that we know that the sequence $\{p_n\}_{n=0}^{\infty}$ converges to some number p of some order α with some asymptotic error constant λ, but we don't know the values of α and λ. The goal of this problem is to develop a method for determining the numerical value of α from the numerical values of the members of the sequence $\{p_n\}_{n=0}^{\infty}$.
Let $E_n := |p_n - p|$ be the error at the nth step of the iteration, and define $\ell_n := \log_{10} E_n$.

(a) Show that for large n, the following approximate identity holds:

$$\ell_n - \alpha\ell_{n-1} \approx \log_{10}\lambda\ .$$

Hint: Just look at the definition of order of convergence.

(b) Using the approximate identity derived in (a) show that

$$\alpha \approx \frac{\ell_n - \ell_{n+1}}{\ell_{n-1} - \ell_n}\ .$$

Note that this approximate formula for α does not depend on the base of the logarithms; if ℓ_n is defined as the log base 10 of E_n, the formula will remain the same.

(c) The data in the table below come from applying the so-called *Newton method* and the *secant method* to find the root of the equation

$$x + \sin x = 1\ ,$$

whose exact value is $p = 0.510973429388569109520013971145080632045358892 62\ldots$. Use the formula derived in part (b) to find empirically the order of convergence α for these two methods.

Table 1: Log_{10} of the errors of the Newton and the secant methods.

n	ℓ_n, Newton	ℓ_n, secant
0	-0.31067	-0.31067
1	-2.85988	-1.49389
2	-7.84087	-2.54052
3	-17.7179	-4.90935
4	-37.4715	-8.33484
5	-76.9787	-14.1282
6	-155.993	-23.3471
7	-314.022	-38.3595
8	-630.079	-62.5907
9	-1262.19	-101.834
10	-2526.42	-165.309
11	-5054.88	-268.027
12	-10111.8	-434.220

Problem 3. As discussed in class, the polynomials of order no higher than n form a linear space with respect to the addition of polynomials and multiplication of a polynomial by a number as follows: if P and Q are polynomials of degree $\leq n$ and $\alpha \in \mathbb{R}$, then the polynomials $P+Q$ and αP are defined as follows:

$$(P+Q)(x) := P(x) + Q(x) \ , \qquad (\alpha P)(x) := \alpha\, P(x) \ .$$

Let $V_n(a, b; w(x))$ stand for the linear space of polynomials defined on the interval with left end a and right end b (at each end, the interval can be open or closed; a and b can be finite or infinite) of degree no greater than n endowed with the inner product

$$(P, Q) = \int_a^b P(x)\, Q(x)\, w(x)\, \mathrm{d}x \ .$$

Dana defined a family of polynomials which she denoted (very modestly!) by D_0, D_1, D_2, ... These polynomials satisfy the following conditions:

(i) the polynomial D_k is of degree k;

(ii) the coefficient of x^k in D_k is equal to 1 (such polynomials are called *monic*, following the definition on page 377 of the book);

(iii) the polynomials D_0, D_1, D_2, ..., D_n form an orthogonal basis in the space of polynomials $V_n(0, \infty; w(x) = \mathrm{e}^{-x})$.

In the solution of this problem the following identity will be handy:

$$\int_0^\infty x^k \, \mathrm{e}^{-x} \, \mathrm{d}x = k!$$

(where, by definition, $0! = 1$).

(a) Clearly, $D_0(x) = 1$ for each $x \in [0, \infty)$. Find the only monic polynomial D_1 of degree 1 that is orthogonal to D_0.

(b) Find the only monic quadratic polynomial D_2 that is orthogonal to both D_0 and D_1.

(c) Show that the polynomial $P(x) = x^2 + 3$ can be represented as a linear combination of the polynomials D_0, D_1 and D_2 as follows: $P = D_2 + 4D_1 + 5D_0$.

(d) Show by direct integration that $(D_0, D_0) = 1$, $(D_1, D_1) = 1$, $(D_2, D_2) = 4$.

(e) Find the orthogonal projection, $\mathrm{proj}_{D_0+2D_1} P$, of the polynomial $P(x) = x^2 + 3$ onto the " straight line"

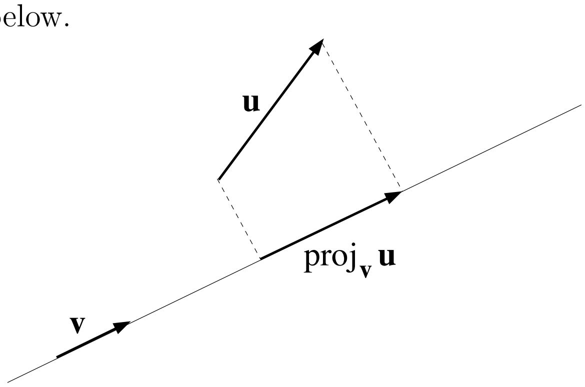
$$\ell := \{t(D_0 + 2D_1) \mid t \in \mathbb{R}\}$$

in the 3-dimensional inner product linear space $V_2(0, \infty; \mathrm{e}^{-x})$. If you have solved part (c), then finding this orthogonal projection should be easy.

Hint: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in the inner product linear space V, then the orthogonal projection of the vector $\mathbf{u}$ onto the straight line in the direction of $\mathbf{v}$ is the vector

$$\mathrm{proj}_{\mathbf{v}} \mathbf{u} = \frac{(\mathbf{u}, \mathbf{v})}{(\mathbf{v}, \mathbf{v})} \, \mathbf{v}$$

– see the picture below.



(f) Finally, let $\tilde{D}_k := \mu_k D_k$, where $\mu_k > 0$ is a constant (depending on k) such that the norm,

$$\left\| \tilde{D}_k \right\| := \sqrt{\left(\tilde{D}_k, \tilde{D}_k \right)} \, ,$$

of the polynomial $\tilde{D}_k$ is 1. Find the explicit expressions for $\tilde{D}_0(x)$, $\tilde{D}_1(x)$, and $\tilde{D}_2(x)$.

Problem 4. Hilbert matrices are a family of matrices that have very high condition numbers. The Hilbert matrix $H^{(n)}$ of size $n \times n$ has matrix elements

$$h_{ij}^{(n)} = \frac{1}{i+j-1} \ .$$

The MATLAB command to generate a Hilbert matrix of size `n` is `hilb(n)`. In this problem you will use MATLAB to study the reliability of the residual and the relative residual as a predictor of the error of an approximate solution of the linear system $H^{(11)}\mathbf{x} = \mathbf{b}$.

(a) In MATLAB, let `A` denote the Hilbert matrix $H^{(11)}$, and `xexact` stand for the exact solution $\mathbf{x}_{\text{exact}} = (\,1 \;\; 2 \;\; 3 \;\; \ldots \;\; 10 \;\; 11\,)^T$. Define the right-hand side $\mathbf{b}$ by typing `b = A*xexact` , and then let `xapprox` be the solution of the linear system $H^{(11)}\mathbf{x} = \mathbf{b}$ found by MATLAB (you can find `xapprox` by using the MATLAB command `inv` to invert the matrix `A`, no need to write your own program).

(b) Find the error $\mathbf{e} = \mathbf{x}_{\text{approx}} - \mathbf{x}_{\text{exact}}$ and the residual $\mathbf{r} = H^{(11)}\mathbf{x}_{\text{approx}} - \mathbf{b}$, and the ℓ_∞-norms of $\mathbf{b}$, $\mathbf{x}_{\text{exact}}$, $\mathbf{e}$, and $\mathbf{r}$, as well as the relative error and the relative residual.

(c) Use the MATLAB command `norm` to find $\|H^{(11)}\|_\infty$, $\|(H^{(11)})^{-1}\|_\infty$, and $\kappa_\infty(H^{(11)})$.

(d) Check that all inequalities proved in the Theorem on page 182 of the book hold in the particular example in this problem. Please write a clear and detailed solution.