Problem 1. Consider the sequence of functions defined by

\[ f_n(x) = \frac{x^n}{n} . \]

(a) Show that \((f_n)\) converges uniformly on \([0, 1]\) and find \(f = \lim f_n\). Show that \(f\) is differentiable and compute \(f'(x)\) for all \(x \in [0, 1]\).

(b) Show that \((f'_n)\) converges on \([0, 1]\). Is the convergence uniform? Set \(g = \lim f'_n\) and compare \(g\) and \(f'\). Are they the same?

Problem 2. Consider the sequence of functions

\[ f_n(x) = \frac{nx + x^2}{2n} , \]

and set \(f(x) = \lim f_n(x)\). Show that \(f\) is differentiable in two ways.

(a) Directly: Compute \(f(x)\) by taking the limit \(n \to \infty\) algebraically, and then find \(f'(x)\).

(b) Using theoretical results: Compute \(f'(x)\) for each \(n \in \mathbb{N}\) and show that the sequence of derivatives \((f'_n)\) converges uniformly on every interval \([-B, B]\). Use some theoretical result (specify which one) to conclude that \(f'(x) = \lim f'_n(x)\).

Problem 3. The result proved in Problem 6 of Homework 3 is called Dini’s Theorem. In this problem you will check that all hypotheses in Dini’s Theorem are necessary.

(a) Consider the sequence of functions \((f_n)\) defined by

\[ f_n(x) = \frac{1}{1 + nx} , \quad x \in (0, 1) . \]

Are they all continuous? Is it true that \(f_{n+1}(x) \leq f_n(x)\) for each \(x \in (0, 1)\)? Does \((f_n)\) converge uniformly? Which condition in Dini’s Theorem is violated for \((f_n)\)?

(b) Define the sequence of functions \((g_n)\) on \([0, 1]\) by

\[ g_n(x) = \begin{cases} 
 n^2 x & \text{for } x \in [0, \frac{1}{n}] , \\
 2n - n^2 x & \text{for } x \in [\frac{1}{n}, \frac{2}{n}] , \\
 0 & \text{for } x \in [\frac{2}{n}, 1] . 
\end{cases} \]

Sketch the graph of \(g_n\). Explain why \(g_n \to 0\) pointwise on \([0, 1]\). Is the sequence \(g_n(x)\) monotone for any \(x \in [0, 1]\)? Is the sequence \((g_n)\) uniformly convergent?
(c) Consider the sequence \((h_n)\) on \([0, 1]\) defined by

\[
h_n(x) = \begin{cases} 
0 & \text{for } x = 0, \\
1 & \text{for } x \in (0, \frac{1}{n}), \\
0 & \text{for } x \in [\frac{1}{n}, 1].
\end{cases}
\]

Does \((h_n)\) converge uniformly? Explain briefly. Which assumption of Dini’s Theorem is not satisfied?

(d) Consider our old friend, the sequence \(r_n(x) = x^n, x \in [0, 1]\). Discuss why we cannot apply Dini’s Theorem to \((r_n)\).

**Problem 4.**

(a) Use the Weierstrass M-test to prove that the function

\[
f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \cdots
\]

is continuous on \([-1, 1]\).

(b) The series

\[
g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots
\]

converges for every \(x \in [-1, 1]\) but does not converge when \(x = 1\). For a fixed \(x_0 \in (-1, 1)\), explain how we can still use the Weierstrass M-test to prove that \(g\) is continuous at \(x_0\).

**Problem 5.** Let

\[
f(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}.
\]

(a) Show that \(f\) is a continuous function on all of \(\mathbb{R}\).

(b) Use the Term-by-term Differentiability Theorem to show that \(f\) is differentiable on any interval \([-B, B]\) for \(B > 0\).

(c) Prove that \(f'\) is continuous on any interval \([-B, B]\) for \(B > 0\).

(d) Do your results from parts (b) and (c) allow you to conclude that \(f'\) exists and is continuous on all of \(\mathbb{R}\)? Justify your answer.
Problem 6. Let $\sum a_n x^n$ be a power series with $a_n \neq 0$, and assume that

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists.

(a) Show that if $L \neq 0$, then the series converges for all $x \in (-\frac{1}{L}, \frac{1}{L})$.

Hint: Recall the proof of the Ratio Test for sequences of numbers.

(b) Show that if $L = 0$, then the series converges for all $x \in \mathbb{R}$.

(c) Show that (a) and (b) continue to hold if $L$ is replaced by the limit

$$L' = \lim_{n \to \infty} s_n , \quad \text{where} \quad s_n = \sup \left\{ \left| \frac{a_{k+1}}{a_k} \right| : k \geq n \right\} .$$

Recalling the definition of limit superior from Problem 5 of Homework 1, we see that $L' = \lim \sup \left| \frac{a_{k+1}}{a_k} \right|$.

Remark: Using the Root Test instead of the Ratio Test, one can easily derive the following formula for the radius of convergence of the power series $\sum a_n x^n$:

$$R = \left( \lim \sup \sqrt[n]{|a_n|} \right)^{-1} .$$

Food for Thought: Abbott, Exercises 6.3.4, 6.4.1, 6.4.7(a,b), 6.5.2.

Hint for Abbott, Exercise 6.5.2: Think about the power series $\sum_{n=0}^{\infty} n! x^n$, $\sum_{n=0}^{\infty} x^n$, $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$, $\sum_{n=1}^{\infty} \frac{x^n}{n!}$.