MATH 4443 **Homework 4** **Due 2/15/18 (Thursday)**

Problem 1. Consider the sequence of functions defined by

$$f_n(x) = \frac{x^n}{n} \ .$$

(a) Show that (f_n) converges uniformly on $[0,1]$ and find $f = \lim f_n$. Show that f is differentiable and compute $f'(x)$ for all $x \in [0,1]$.

(b) Show that (f_n') converges on $[0,1]$. Is the convergence uniform? Set $g = \lim f_n'$ and compare g and f'. Are they the same?

Problem 2. Consider the sequence of functions

$$f_n(x) = \frac{nx + x^2}{2n} \ ,$$

and set $f(x) = \lim f_n(x)$. Show that f is differentiable in two ways.

(a) *Directly:* Compute $f(x)$ by taking the limit $n \to \infty$ algebraically, and then find $f'(x)$.

(b) *Using theoretical results:* Compute $f'(x)$ for each $n \in \mathbb{N}$ and show that the sequence of derivatives (f_n') converges uniformly on every interval $[-B, B]$. Use some theoretical result (specify which one) to conclude that $f'(x) = \lim f_n'(x)$.

Problem 3. The result proved in Problem 6 of Homework 3 is called *Dini's Theorem.* In this problem you will check that all hypotheses in Dini's Theorem are necessary.

(a) Consider the sequence of functions (f_n) defined by

$$f_n(x) = \frac{1}{1+nx} \ , \qquad x \in (0,1) \ .$$

Are they all continuous? Is it true that $f_{n+1}(x) \leq f_n(x)$ foe each $x \in (0,1)$? Does (f_n) converge uniformly? Which condition in Dini's Theorem is violated for (f_n)?

(b) Define the sequence of functions (g_n) on $[0,1]$ by

$$g_n(x) = \begin{cases} n^2 x & \text{for } x \in [0, \frac{1}{n}] \ , \\ 2n - n^2 x & \text{for } x \in [\frac{1}{n}, \frac{2}{n}] \ , \\ 0 & \text{for } x \in [\frac{2}{n}, 1] \ . \end{cases}$$

Sketch the graph of g_n. Explain why $g_n \to 0$ pointwise on $[0,1]$. Is the sequence $g_n(x)$ monotone for any $x \in [0,1]$? Is the sequence (g_n) uniformly convergent?

(c) Consider the sequence (h_n) on $[0,1]$ defined by

$$h_n(x) = \begin{cases} 0 & \text{for } x = 0 \ , \\ 1 & \text{for } x \in (0, \frac{1}{n}) \ , \\ 0 & \text{for } x \in [\frac{1}{n}, 1] \ . \end{cases}$$

Does (h_n) converge uniformly? Explain briefly. Which assumption of Dini's Theorem is not satisfied?

(d) Consider our old friend, the sequence $r_n(x) = x^n$, $x \in [0,1]$. Discuss why we cannot apply Dini's Theorem to (r_n).

Problem 4.

(a) Use the Weierstrass M-test to prove that the funciton

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \cdots$$

is continuous on $[-1,1]$.

(b) The series

$$g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

converges for every $x \in [-1,1)$ but does not converge when $x = 1$. For a fixed $x_0 \in (-1,1)$, explain how we can still use the Weierstrass M-test to prove that g is continuous at x_0.

Problem 5. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2} \ .$$

(a) Show that f is a continuous function on all of $\mathbb{R}$.

(b) Use the Term-by-term Differentiability Theorem to show that f is differentiable on any interval $[-B, B]$ for $B > 0$.

(c) Prove that f' is continuous on any interval $[-B, B]$ for $B > 0$.

(d) Do your results from parts (b) and (c) allow you to conclude that f' exists and is continuous on all of $\mathbb{R}$? Justify your answer.

Problem 6. Let $\sum a_n x^n$ be a power series with $a_n \neq 0$, and assume that

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists.

(a) Show that if $L \neq 0$, then the series converges for all $x \in \left(-\frac{1}{L}, \frac{1}{L}\right)$.

Hint: Recall the proof of the Ratio Test for sequences of numbers.

(b) Show that if $L = 0$, then the series converges for all $x \in \mathbb{R}$.

(c) Show that (a) and (b) continue to hold if L is replaced by the limit

$$L' = \lim_{n\to\infty} s_n , \qquad \text{where} \qquad s_n = \sup \left\{ \left| \frac{a_{k+1}}{a_k} \right| \ : \ k \geq n \right\} .$$

Recalling the definition of limit superior from Problem 5 of Homework 1, we see that $L' = \limsup \left| \frac{a_{k+1}}{a_k} \right|$.

Remark: Using the Root Test instead of the Ratio Test, one can easily derive the following formula for the radius of convergence of the power series $\sum a_n x^n$:

$$R = \left(\limsup \sqrt[n]{|a_n|} \right)^{-1} .$$

Food for Thought: Abbott, Exercises 6.3.4, 6.4.1, 6.4.7(a,b), 6.5.2.

Hint for Abbott, Exercise 6.5.2: Think about the power series $\sum_{n=0}^{\infty} n! x^n$, $\sum_{n=0}^{\infty} x^n$, $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$, $\sum_{n=1}^{\infty} \frac{x^n}{n}$, $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$, $\sum_{n=1}^{\infty} \frac{x^n}{n!}$.