MATH 4093/5093 Homework 4 Due Mon, 10/04/10

Problem 1. The explicit 4-step multistep method

$$\frac{w_{i+1} - w_{i-3}}{h} = \frac{4}{3}\left[2f(t_i, w_i) - f(t_{i-1}, w_{i-1}) + 2f(t_{i-2}, w_{i-2})\right]$$

is know as *Milne's method*. It is easy to show that is of order 4 (this is done on page 594 of the book). The 4-step Adams-Bashforth (AB4) method is given on page 585 of the book; it is also of order 4.

(a) Prove that AB4 is strongly stable.

(b) Prove that Milne's method is weakly stable.

(c) Download the code `ab4.m` from `http://www.pcs.cnu.edu/~bbradie/mivps.html`, and create the file `milne.m` which is almost the same as `ab4.m`, except that in the first line the name of the function should be changed to `milne`, and the line

```
x0 = x0 + h/24 * ( 55*fnew - 59*oldf(3,:) + 37*oldf(2,:) - 9*oldf(1,:) );
```

should be replaced by

```
x0 = wi(1:neqn,i-3) + 4*h/3 * ( 2*fnew - oldf(3,:) + 2*oldf(2,:) );
```

Compare Milne's method and AB4 for solving the initial-value problem

$$y'(t) + y(t) = -\mathrm{e}^{-t}\sin t\ , \quad y(0) = 1\ .$$

Use stepsize $h = 0.1$ and compute the approximate solution over the interval $t \in [0, 10]$. Plot both functions on the same plot, together with the exact solution, $y(t) = \mathrm{e}^{-t}\cos t$. Attach the plot. Explain what you observe.

(d) Show that if the stepsize h is reduced to 0.005, but the interval is extended to $t \in [0, 15]$, then the sawtooth oscillations still appear in the approximate solution obtained from Milne's method. Attach the plot of the two approximate solutions (Milne's and AB4) and the theoretical solution.

Problem 2. Suppose that we want to construct a variable step size algorithm from the following two methods: the third-order method

$$\tilde{w}_{i+1} = w_i + \frac{1}{4}k_1 + \frac{3}{8}k_2 + \frac{3}{8}k_3$$

is used to approximate the local truncation error in the second-order method

$$w_{i+1} = w_i + \frac{1}{4}k_1 + \frac{3}{4}k_2\ ,$$

where

$$\begin{aligned} k_1 &= hf\left(t_i, w_i\right) \ , \\ k_2 &= hf\left(t_i + \frac{2}{3}h, w_i + \frac{2}{3}k_1\right) \ , \\ k_3 &= hf\left(t_i + \frac{2}{3}h, w_i + \frac{2}{3}k_2\right) \ . \end{aligned}$$

(a) In terms of k_1, k_2 and k_3, what is the formula for the local truncation error estimate? *Hint:* See the text on pages 611–612 of the book, discussing the RKF45 method.

(b) What is the formula for the step size adjustment factor q?

Problem 3. Download the Runge-Kutta-Fehlberg order 4–order 5 code `rkf45.m` from `http://www.pcs.cnu.edu/~bbradie/mivps.html`.

(a) Modify `rkf45.m` to make it save the stepsize h_i at each of the values of t_i. The first line of your code should look like this:

```
function [wi, ti, hi, count] = rkf45_modified ( RHS, t0, x0, tf, parms)
```

Attach a printout of your code.

(b) Use your code to solve the initial value problem

$$y'(t) + 50y(t) = 50\cos t \ , \quad t \in [0, 10] \ , \quad y(0) = 1 \ ,$$

with $h_{\min} = 0.001$, $h_{\max} = 0.075$, and different values of the absolute error tolerance, namely, TOL $= 10^{-2}$, 10^{-4}, 10^{-6}, and 10^{-8}. (In the case TOL $= 10^{-2}$, the vector of parameters `parms` should look like this: `[0.001,0.075,1e-2]`). Plot the stepsize h_i as a function of t_i for each of these four values of TOL, on the same graph. On your printout, clearly mark what symbols correspond to which value of TOL. Discuss briefly what you see. Do the graphs look reasonable? Why?

Problem 4. Directly from the definition on page 23 of the book, find the rates of convergence α and the asymptotic error constants λ for the sequences (all of which tend to 0)

$$(a) \ \ p_n = \frac{17}{n^5} \ ; \qquad (b) \ \ p_n = 3^{-n} \ ; \qquad (c) \ \ p_n = 10^{-5^n} \ .$$

Problem 5. Let the sequence $\{p_n\}_{n=1}^{\infty}$ be defined by $p_0 = 0$, $p_n = 1 - \frac{1}{2}\sin p_{n-1}$ for $n \geq 1$.

(a) Think of the sequence $\{p_n\}_{n=0}^{\infty}$ as a functional iteration, $p_n = g(p_{n-1})$, for an appropriate function g. Write g explicitly. Use the Theorem on page 84 of the book to prove that g has a fixed point in the interval $[0, \frac{\pi}{2}]$.

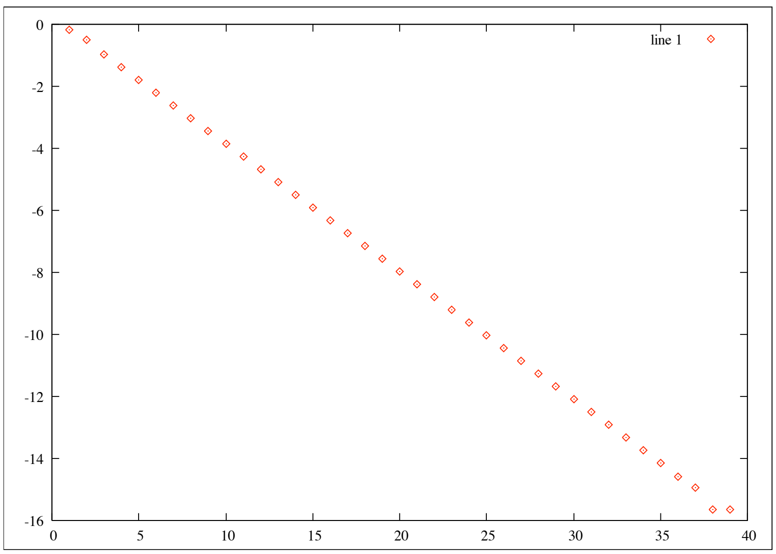
Hint: You have to show that $g([0, \frac{\pi}{2}])$ is a subset of $[0, \frac{\pi}{2}]$. One way of doing this is to check that g is continuous (which will be quite clear once you obtain the explicit form of g), then check that both $g(0)$ and $g(\frac{\pi}{2})$ are in the interval $[0, \frac{\pi}{2}]$, and finally show that g is monotone in $[0, \frac{\pi}{2}]$, which implies that all values of $g(x)$ for $x \in [0, \frac{\pi}{2}]$ are between $g(0)$ and $g(\frac{\pi}{2})$. To check the monotonicity of g, find $g'(x)$ and show that g' does not change its sign in the interval $(0, \frac{\pi}{2})$.

(b) Discuss the uniqueness of the fixed point using the Theorem on page 84.

(c) The fixed point of the function g in $[0, \frac{\pi}{2}]$ is $p = 0.684036656677830\ldots$. I defined in Matlab the variable `p` with this value, and then ran the following one-line code:

```
error=[]; x = 0.0; for n=1:39 error=[error abs(x-p)]; x=1-sin(x)/2; end;
```

This created the array `error` containing the absolute values of the errors, $|e_n| = |p_n - p|$. Then the Matlab command `plot(log(error)/log(10),'o')` plotted $\log_{10}|e_n|$ versus n; the plot is shown in the figure below.



Convince me that the fact that we see a straight line in this graph means that the iteration converges linearly (i.e., has order of convergence $\alpha = 1$), and find the value of the asymptotic error constant λ. You can use the following values: $|e_{19}| \approx 1.07858866 \times 10^{-8}$, $|e_{20}| \approx 4.17968304 \times 10^{-9}$, $|e_{21}| \approx 1.61968594 \times 10^{-9}$ (you will need only two of these values, but I gave you three of them to double-check your method).

Hint: For large enough values of n, the Definition of α and λ on page 23 implies that $|e_n| \approx \lambda |e_n|^{\alpha}$. Take logarithms (say, base 10, as in the picture) of both sides of this equality. Now think about the slope of the approximate straight line in the figure: if $(n, \log_{10}|e_n|)$ and $(n+1, \log_{10}|e_{n+1}|)$ are two adjacent points, then what is the slope of the straight line connecting them? How can you get α and λ from this slope?