# MATH 5163 Homework 4 Due Thu, 2/13/14

**Problem 1. [Test functions]**

Let $\phi \in \mathscr{D}(\mathbb{R})$. Consider the following sequences of functions:

$$\text{(a)}\quad \psi_k(x) := \frac{1}{k}\,\phi(x)\ ; \qquad \text{(b)}\quad \rho_k(x) := \frac{1}{k}\,\phi(kx)\ ; \qquad \text{(c)}\quad \sigma_k(x) := \frac{1}{k}\,\phi\left(\frac{x}{k}\right)\ .$$

For each sequence explain if it converges in $\mathscr{D}(\mathbb{R})$ as $k \to \infty$ – if it does, to what limiting function; if it does not, why.

*Hint:* Only $\psi_k$ converges.

**Problem 2. [Convolution with a mollifier]**

Recall that the function $\eta : \mathbb{R}^n \to \mathbb{R}$ was defined in class as

$$\eta(\mathbf{x}) := \begin{cases} C\exp\left(\dfrac{1}{|\mathbf{x}|^2 - 1}\right)\ , & |\mathbf{x}| < 1\ , \\ 0\ , & |\mathbf{x}| \geq 1\ , \end{cases} \tag{1}$$

where the constant $C$ is such that $\|\eta\|_{L^1(\mathbb{R}^n)} = 1$. For any $\varepsilon > 0$, define $\eta_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by

$$\eta_\varepsilon(\mathbf{x}) := \frac{1}{\varepsilon^n}\,\eta\left(\frac{\mathbf{x}}{\varepsilon}\right)\ . \tag{2}$$

It is easy to show that $\eta_\varepsilon \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}\eta_\varepsilon = \overline{B(\mathbf{0},\varepsilon)}$, $\eta_\varepsilon \geq 0$, $\|\eta_\varepsilon\|_{L^1(\mathbb{R}^n)} = 1$.

Let $A \subset \mathbb{R}^n$ be a bounded open subset of $\mathbb{R}^n$ (the openness of $A$ is not essential), and let $f := \chi_A : \mathbb{R}^n \to \mathbb{R}$ be the indicator function of the set $A$, i.e.,

$$f(\mathbf{x}) := \begin{cases} 1\ , & \mathbf{x} \in A\ , \\ 0\ , & \mathbf{x} \notin A\ . \end{cases}$$

Define $f_\varepsilon := f * \eta_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ as the convolution of $f$ and $\eta_\varepsilon$:

$$f_\varepsilon(\mathbf{x}) := (f * \eta_\varepsilon)(\mathbf{x}) := \int_{\mathbb{R}^n} f(\mathbf{y})\,\eta_\varepsilon(\mathbf{x} - \mathbf{y})\,\mathrm{d}\mathbf{y}\ .$$

(a) Show that if $\mathbf{x} \in \mathbb{R}^n$ is such that $B(\mathbf{x},\varepsilon) \subset A$, then $f_\varepsilon(\mathbf{x}) = 1$.

(b) What is the value of $f_\varepsilon(\mathbf{x})$ if $\mathbf{x} \in \mathbb{R}^n$ is such that $B(\mathbf{x},\varepsilon) \cap A = \varnothing$? Justify your claim.

(c) How differentiable is $f_\varepsilon$? (Recall what you know about differentiation of a convolution; there is no need of a detailed proof.)

(d) Does $f_\varepsilon$ belong to $\mathscr{D}(\mathbb{R}^n)$? Explain briefly.

**Problem 3. [The space of distributions $\mathscr{D}'(\mathbb{R})$]**
Show that $u(x) := \mathrm{e}^x \in \mathscr{D}'(\mathbb{R})$, in the sense that it defines a continuous linear functional on $\mathscr{D}(\mathbb{R})$ by

$$\langle u, \phi \rangle := \int \mathrm{e}^x \, \phi(x) \, \mathrm{d}x \ , \qquad \phi \in \mathscr{D}(\mathbb{R}) \ .$$

**Problem 4. [Convergence in $\mathscr{D}'(\mathbb{R})$]**
Prove that $\delta_k \to 0$ in $\mathscr{D}'(\mathbb{R})$ as $k \to \infty$.

**Problem 5. [Delta-like sequences in $\mathscr{D}'(\mathbb{R})$]**
The *Weighted Mean Value Theorem for Integrals* states that if the function $f : [a,b] \to \mathbb{R}$ is continuous on the interval $[a,b]$ and the function $g : [a,b] \to \mathbb{R}$ is integrable and does not change sign on $[a,b]$, then there exists a number $c \in (a,b)$ such that

$$\int_a^b f(x) \, g(x) \, \mathrm{d}x = f(c) \int_a^b g(x) \, \mathrm{d}x \ .$$

Use this theorem to prove that $\eta_\varepsilon \to \delta$ in $\mathscr{D}'(\mathbb{R})$ as $\varepsilon \to 0^+$, where $\eta_\varepsilon : \mathbb{R} \to \mathbb{R}$ is defined as in (1) and (2) (but for $n = 1$).

*Remark 1:* Instead of using the above theorem, one can use the continuity of the test function $\phi \in \mathscr{D}(\mathbb{R})$. Namely, the continuity of $\phi$ implies that for any $\mu > 0$ there exists a number $\varepsilon_0 > 0$ such that $|\phi(x) - \phi(0)| < \mu$ for any $x \in (-\varepsilon_0, \varepsilon_0)$. Using the properties of $\eta_\varepsilon$, we obtain that for all $\varepsilon \leq \varepsilon_0$

$$|\langle \eta_\varepsilon, \phi \rangle - \phi(0)| = \left| \int \eta_\varepsilon(x) \, \phi(x) \, \mathrm{d}x - \phi(0) \right| \leq \int \eta_\varepsilon(x) \, |\phi(x) - \phi(0)| \, \mathrm{d}x < \mu \int \eta_\varepsilon(x) \, \mathrm{d}x = \mu \ ,$$

which implies that $\langle \eta_\varepsilon, \phi \rangle \to \phi(0) = \langle \delta, \phi \rangle$ for any $\phi \in \mathscr{D}(\mathbb{R})$ as $\varepsilon \to 0^+$, i.e., that $\eta_\varepsilon \to \delta$ in $\mathscr{D}(\mathbb{R})$ as $\varepsilon \to 0^+$.

*Remark 2:* One can also prove that the functions $\frac{1}{2\varepsilon} \chi_{[-\varepsilon,\varepsilon]}(x)$, $\frac{\varepsilon}{\pi(x^2+\varepsilon^2)}$, $\frac{1}{2\sqrt{\pi\varepsilon}} \mathrm{e}^{-x^2/(4\varepsilon)}$, $\frac{1}{\pi x} \sin \frac{x}{\varepsilon}$, $\frac{1}{\pi x^2} \sin^2 \frac{x}{\varepsilon}$ converge to $\delta$ in $\mathscr{D}'(\mathbb{R})$ as $\varepsilon \to 0^+$.

**Problem 6. [“Naive” definition of $\delta_a'$]**
Recall that the $k$th derivative, $\delta_a^{(k)}(x) := \frac{\mathrm{d}^k}{\mathrm{d}x^k} \delta_a(x)$, of $\delta_a(x)$ is defined by the formula $\langle \delta_a^{(k)}, \phi \rangle := (-1)^k \, \phi^{(k)}(a)$ for any $\phi \in \mathscr{D}(\mathbb{R})$, which can be written formally as

$$\int \delta_a^{(n)}(x) \, \phi(x) \, \mathrm{d}x := (-1)^n \, \phi^{(n)}(a) \ . \tag{3}$$

The motivation for this definition came from treating the derivatives of $\delta_a(x)$ as ordinary functions, integrating by parts, and using that at $\pm\infty$ the test function $\phi$ is equal to zero.

In this problem you will give a meaning of the formal definition of a derivative of $\delta_a(x)$ that looks like the derivative of an "ordinary" function:

$$\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x}\,\delta_a(x) \quad \text{“:=”} \quad \lim_{h\to 0}\frac{\delta_a(x+h)-\delta_a(x)}{h}\ ; \tag{4}$$

here the tilde over the derivative sign simply means that this definition is different from the definition (3) of the derivative of $\delta_a(t)$. Inspired by (4), define

$$\left\langle \frac{\widetilde{\mathrm{d}}}{\mathrm{d}x}\,\delta_a(x), \phi \right\rangle \equiv \int \left(\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x}\,\delta_a(x)\right)\phi(x)\,\mathrm{d}x := \lim_{h\to 0}\int \frac{\delta_a(x+h)-\delta_a(x)}{h}\,\phi(x)\,\mathrm{d}x\ . \tag{5}$$

(a) Change the variable $x$ in $\int \delta_a(x+h)\,\phi(x)\,\mathrm{d}x$ to $z = x+h$ to compute this integral.

(b) Using your result from part (a), find $\displaystyle\int \frac{\delta_a(x+h)-\delta_a(x)}{h}\,\phi(x)\,\mathrm{d}x$.

(c) Find $\displaystyle\int \left(\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x}\,\delta_a(x)\right)\phi(x)\,\mathrm{d}x$ defined by (5), and compare your result with the definition of $\delta_a'$ given by equation (3). Discuss briefly your findings.

**Problem 7. [A product of a $C^\infty(\mathbb{R})$ function with $\delta$, $\delta'$, and $\delta''$]**

In all parts of this problem assume that $f \in C^\infty(\mathbb{R})$.

(a) Show that the product $f\,\delta$ defined for an arbitrary test function $\phi$ as

$$\langle f\,\delta, \phi\rangle := \int f(x)\,\delta(x)\,\phi(x)\,\mathrm{d}x$$

is a distribution in $\mathscr{D}'(\mathbb{R})$ and

$$f(x)\,\delta(x) = f(0)\,\delta(x)\ .$$

*Hint:* Show that $\langle f\,\delta, \phi\rangle = f(0)\,\phi(0)$ for any test function $\phi$.

(b) Prove the equality

$$f(x)\,\delta'(x) = -f'(0)\,\delta(x) + f(0)\,\delta'(x)\ .$$

(c) Find an expression for $f(x)\,\delta''(x)$ like in part (b) (i.e., an expression does not involve the function $f(x)$ and its derivatives but only the value of $f$ and its derivatives at 0).

**Problem 8. [Derivative of a function with a "corner"]**

Consider the function

$$l(x) := \begin{cases} 0\ , & x < 0\ , \\ x\ , & x \geq 0 \end{cases}$$

as an element of $\mathscr{D}'(\mathbb{R})$ and show that it is differentiable and $l'$ is the Heaviside function $H$.

*Hint:* Imitate the proof that $H' = \delta$ that was given in class.