MATH 5403 Homework 4 Due Thu, 11/14/19

Problem 1. [Noether's Theorem]

Consider the action
$$J[y] = \int_{x_0}^{x_1} x\,(y')^2 \,\mathrm{d}x \; . \tag{1}$$
Consider the following transformations of x and y:
$$x^* = \Phi(x, y; \varepsilon) = \exp(\mathrm{e}^{2\varepsilon} \ln x) \; , \qquad y^* = \Psi(x, y; \varepsilon) = y\mathrm{e}^{\varepsilon} \; . \tag{2}$$

(a) Prove that the transformations (2) form a 1-parameter group, i.e., that
$$\Phi\big(\Phi(x, y; \varepsilon), \Psi(x, y; \varepsilon); \nu\big) = \Phi(x, y; \varepsilon + \nu) \; ,$$
$$\Psi\big(\Phi(x, y; \varepsilon), \Psi(x, y; \varepsilon); \nu\big) = \Psi(x, y; \varepsilon + \nu) \; .$$

(b) Prove that the transformation (2) is a *variational symmetry* of the action (1), i.e., that for any subinterval $[a, b]$ of $[x_0, x_1]$, we have
$$\int_a^b f\big(x, y(x), y'(x)\big) \,\mathrm{d}x = \int_{\Phi(a,y(a);\varepsilon)}^{\Phi(b,y(b);\varepsilon)} f\big(x^*, y^*(x^*), (y^*)'(x^*)\big) \,\mathrm{d}x^* \; .$$
To this end, you have to show that
$$f\big(x, y(x), y'(x)\big) \,\mathrm{d}x = f\big(x^*, y^*(x^*), (y^*)'(x^*)\big) \,\mathrm{d}x^* \; ,$$
which for the action (1) reads
$$x^* \left(\frac{\mathrm{d}y^*}{\mathrm{d}x^*}\right)^2 \mathrm{d}x^* = x \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \mathrm{d}x \; .$$
To compute $\frac{\mathrm{d}y^*}{\mathrm{d}x^*}$, you may use the chain rule: $\frac{\mathrm{d}y^*}{\mathrm{d}x^*} = \frac{\mathrm{d}y^*}{\mathrm{d}y} \frac{\mathrm{d}y}{\mathrm{d}x} \frac{\mathrm{d}x}{\mathrm{d}x^*}$.

(c) If a transformation is a variational symmetry for the action
$$J[y] = \int_{x_0}^{x_1} f(x, y, y') \,\mathrm{d}x \; .$$
then Noether's Theorem claims that the quantity
$$A(x, y, y') := \frac{\partial f}{\partial y'} \psi + \left(f - y' \frac{\partial f}{\partial y'}\right) \phi \; ,$$
is constant along any extremal of the action $J[y]$, where
$$\phi(x, y) := \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \Phi(x, y; \varepsilon)\bigg|_{\varepsilon=0} \quad \text{and} \quad \psi(x, y) := \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \Psi(x, y; \varepsilon)\bigg|_{\varepsilon=0}$$
are the *generators* of the transformation. Compute the generators and find the conserved quantity $A(x, y, y')$ for the action (1) and the transformation (2).

(d) Write down the Euler-Lagrange equation for the action (1); do not solve them yet. Use them to show that $\frac{\mathrm{d}}{\mathrm{d}t} A(t, y(t), y'(t)) = 0$ along an extremal of the action. This will be very easy if you notice that the conserved quantity can be written as

$$A(x, y, y') = 2(xy')[y - (xy') \ln x]$$

(differentiate this expression by first applying the product rule; do not separate (xy')).

(e) Now solve the Euler-Lagrange equation and plug your solution in the concrete expression for $A(x, y, y')$ to show again that $A(x, y, y')$ is constant along your solution.

Problem 2. [Eigenfunctions as constrained extremals]
In this problem you will find the extremals of the integral

$$J[y] = \int_0^\pi (y')^2 \, \mathrm{d}x$$

subject to the boundary conditions

$$y(0) = 0 \ , \quad y(\pi) = 0 \ ,$$

and the constraint

$$\int_0^\pi y^2 \, \mathrm{d}x = 1 \ .$$

(a) Show that the Euler-Lagrange equation for the function $F := f - \lambda g$ is $y'' + \lambda y = 0$.

(b) Assume that λ is strictly negative, set $\lambda = -\alpha^2$, where (without loss of generality) $\alpha > 0$, and write down the general solution of the Euler-Lagrange equation in this case (it will be a sum of two real exponents). Show that when you impose the boundary conditions on your solution, you will obtain that $y(x) \equiv 0$, so the constraint cannot be satisfied.

(c) Assume that $\lambda = 0$ and repeat the steps from part (b) to show that in this case the problem again has no solution.

(d) Consider the only remaining case, $\lambda = \alpha^2$. Show that, for the solution to be not identically zero, the constant α must be a positive integer:

$$\alpha = k \in \mathbb{N} \ .$$

Impose the constraint to find the constrained extremals.

Problem 3. [Physical interpretation of the constraint forces]

Consider a point particle of mass m in $\mathbb{R}^n$ that is moving in the field of the potential force $\mathbf{F} = -\nabla U(\mathbf{q})$, $\mathbf{q} \in \mathbb{R}^n$. We assume that the system is autonomous, i.e., its Lagrangian does not contain the time explicitly:

$$L(\mathbf{q}, \dot{\mathbf{q}}) = K(\mathbf{q}, \dot{\mathbf{q}}) - U(\mathbf{q}) \ ,$$

where $K(\mathbf{q}, \dot{\mathbf{q}})$ and $U(t, \mathbf{q})$ are the kinetic and the potential energy of the particle; the dot stands for the time derivative. We also assume that the particle is subjected to a scleronomic (i.e., time-independent) holonomic constraint

$$g(\mathbf{q}) = 0 \ , \tag{3}$$

i.e., it can only move in the $(n-1)$-dimensional *constraint manifold*, $C := \{\mathbf{q} \in \mathbb{R}^n : g(\mathbf{q}) = 0\}$; assume, as usual, that $\nabla g \neq 0$ on C.

In this problem $\mathbf{q} = (q_1, \dots, q_n)$ will stand for *any* generalized coordinates, while the notation $\mathbf{y} = (y_1, \dots, y_n)$ will mean the Cartesian coordinates, in which the Lagrangian has the form

$$L(\mathbf{y}, \dot{\mathbf{y}}) = \frac{m}{2} |\dot{\mathbf{y}}|^2 - U(\mathbf{y}) \ . \tag{4}$$

Recall that the Lagrange multiplier algorithm instructs us to define the function

$$F(\mathbf{q}, \dot{\mathbf{q}}) = L(\mathbf{q}, \dot{\mathbf{q}}) - \lambda(t) g(\mathbf{q}) \ , \tag{5}$$

and to solve the Euler-Lagrange equations,

$$\frac{\partial F}{\partial \mathbf{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial F}{\partial \dot{\mathbf{q}}} = 0 \ , \qquad \text{or, equivalently,} \qquad \frac{\partial L}{\partial \mathbf{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\mathbf{q}}} = \lambda(t) \, \nabla g(\mathbf{q}) \ , \tag{6}$$

for the function $F(\mathbf{q}, \dot{\mathbf{q}})$ (5) together with the constraint equation (3) in order to find the functions $\mathbf{q}(t)$ and $\lambda(t)$ (a total of $(n+1)$ unknown functions).

(a) Write down the Euler-Lagrange equations (6) for the function $F(\mathbf{q}, \dot{\mathbf{q}})$ (5) for the Lagrangian written in Cartesian coordinates, (4), and compare with Newton's second law, $m\mathbf{a} = \mathbf{F}_{\text{net}}$, to interpret the term $\mathbf{N} := \lambda(t) \nabla g(\mathbf{y})$ as a *constraint force*, i.e., a force that the constraint (3) exerts on the particle.

(b) Use the fact that the particle always has to belong to the constraint manifold C (i.e., must satisfy the constraint (3) at any moment t), to prove the identities

$$\dot{\mathbf{q}} \cdot \nabla g(\mathbf{q}) = 0 \ , \tag{7}$$

and

$$\ddot{\mathbf{q}} \cdot \nabla g(\mathbf{q}) = -\dot{\mathbf{q}} \cdot \operatorname{Hess} g(\mathbf{q}) \cdot \dot{\mathbf{q}} \ , \tag{8}$$

where $\operatorname{Hess} g(\mathbf{q}) := \left(\dfrac{\partial^2 g}{\partial y_i \, \partial y_j}(\mathbf{q}) \right)$ is the *Hessian* of the function g, i.e., the $n \times n$ matrix of its second partial derivatives (which is automatically symmetric), and

$$\mathbf{a} \cdot \operatorname{Hess} g(\mathbf{q}) \cdot \mathbf{b} := \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \, \frac{\partial^2 g}{\partial y_i \, \partial y_j}(\mathbf{q}) \, b_i \ .$$

(c) Define the *energy* (in generalized coordinates),

$$E(t) := \dot{\mathbf{q}} \cdot \frac{\partial L}{\partial \dot{\mathbf{q}}} - L \ , \qquad \text{where} \qquad \dot{\mathbf{q}} \cdot \frac{\partial L}{\partial \dot{\mathbf{q}}} := \sum_{i=1}^{n} \dot{q}_i \, \frac{\partial L}{\partial \dot{q}_i} \ . \tag{9}$$

Use the Euler-Lagrange equations (6) and some of the identities obtained above (in generalized coordinates $\mathbf{q}$) to show that the constraint force $\mathbf{N}$ does not do any work, i.e., the energy is conserved despite the presence of a constraint.

Physically, this means that $\mathbf{N}$ is always perpendicular to the velocity of the particle, i.e., orthogonal to the constraint manifold C. Recalling that the $\nabla g(\mathbf{q})$ is perpendicular to C at each point $\mathbf{q} \in C$ (cf. (7)), we conclude that $\mathbf{N}$ is collinear with $\nabla g(\mathbf{q})$.

(d) In Cartesian coordinates the Euler-Lagrange equations read

$$m\ddot{\mathbf{y}} = \mathbf{F} + \mathbf{N} \ , \tag{10}$$

where $\mathbf{F} = -\nabla U(\mathbf{y})$ and $\mathbf{N} = \lambda(t) \nabla g(\mathbf{y})$. Multiply equation (10) by $\dot{\mathbf{y}}$ to rederive the conservation of energy (where the energy is expressed in Cartesian coordinates as $E = \frac{m}{2}|\dot{\mathbf{y}}|^2 + U(\mathbf{y})$). Point out what previously obtained facts you are using in your derivation.

(e) Multiply (10) by ∇g to obtain the expression

$$\lambda |\nabla g| = -\frac{m}{|\nabla g|} (\dot{\mathbf{y}} \cdot \operatorname{Hess} g(\mathbf{y}) \cdot \dot{\mathbf{y}}) - \frac{\mathbf{F} \cdot \nabla g}{|\nabla g|} \ .$$

To elucidate the physical interpretation of the forces acting on the particle, one can decompose the force $\mathbf{F}$ into a component $\mathbf{F}_\perp = \dfrac{(\mathbf{F} \cdot \nabla g)}{|\nabla g|^2} \nabla g$ perpendicular to C, and a component $\mathbf{F}_\parallel = \mathbf{F} - \mathbf{F}_\perp$ parallel to C. With these notations, equation (10) becomes $m\ddot{\mathbf{y}} = \mathbf{F}_\parallel + (\mathbf{F}_\perp + \lambda \nabla g)$.

(f) In the rest of the problem we want to relate the component of the constraint force containing $\dot{\mathbf{y}} \cdot \operatorname{Hess} g(\mathbf{y}) \cdot \dot{\mathbf{y}}$ to the geometry of the constraint manifold and the motion of the particle. To make things easier to visualize, assume that $n = 2$ and denote the Cartesian coordinates in $\mathbb{R}^2$ by (y, z). Let the constraint manifold be given as a parameterized curve in $\mathbb{R}^2$:

$$C = \left\{ \mathbf{Y}(\xi) = (Y(\xi), Z(\xi)) \in \mathbb{R}^2 : \xi \in \mathbb{R} \right\} \ . \tag{11}$$

The vector $\mathbf{Y}'(\xi) := \frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}\xi}(\xi)$ is tangent to C. The curvature of the curve C measures the rate of rotation of the tangent vector to C as the point moves along C. To eliminate the dependence on the particular choice of parameterization, we parameterize C by its natural parameter, the arclength s, given by $\frac{\mathrm{d}s}{\mathrm{d}\xi} = |\mathbf{Y}'(\xi)|$. Using the parameter s, we have $\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}s} = \frac{\mathrm{d}\xi}{\mathrm{d}s}\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}\xi} = \frac{1}{\mathrm{d}s/\mathrm{d}\xi}\mathbf{Y}'(\xi) = \frac{\mathbf{Y}'(\xi)}{|\mathbf{Y}'(\xi)|}$. The rate of rotation of this vector (with respect to s) is $\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}s} = \frac{\mathrm{d}\xi}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathbf{Y}'(\xi)}{|\mathbf{Y}'(\xi)|} = \frac{1}{\mathrm{d}s/\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathbf{Y}'(\xi)}{|\mathbf{Y}'(\xi)|} = \frac{1}{|\mathbf{Y}'(\xi)|}\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathbf{Y}'(\xi)}{|\mathbf{Y}'(\xi)|}$ which, with the help of

$$\frac{\mathrm{d}}{\mathrm{d}\xi}|\mathbf{Y}'(\xi)| = \frac{\mathrm{d}}{\mathrm{d}\xi}\sqrt{\mathbf{Y}'(\xi)\cdot\mathbf{Y}'(\xi)} = \frac{\mathbf{Y}'(\xi)\cdot\mathbf{Y}''(\xi)}{|\mathbf{Y}'(\xi)|}\ ,$$

can be written as

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}s} = \frac{1}{|\mathbf{Y}'(\xi)|}\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathbf{Y}'(\xi)}{|\mathbf{Y}'(\xi)|} = \frac{\big||\mathbf{Y}'(\xi)|^2\mathbf{Y}''(\xi) - (\mathbf{Y}'(\xi)\cdot\mathbf{Y}''(\xi))\mathbf{Y}'(\xi)\big|}{|\mathbf{Y}'(\xi)|^4}\ ,$$

or, after tedious algebra, as

$$\text{curvature} = \lim\frac{\Delta\phi}{\Delta s} = \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}s} = \frac{|Y'(\xi)Z''(\xi) - Y''(\xi)Z'(\xi)|}{[Y'(\xi)^2 + Z'(\xi)^2]^{3/2}}\ ,$$

where $Y(\xi)$ and $Z(\xi)$ are the functions from (11). The radius of curvature is the reciprocal of this expression:

$$\rho = \frac{[Y'(\xi)^2 + Z'(\xi)^2]^{3/2}}{|Y'(\xi)Z''(\xi) - Y''(\xi)Z'(\xi)|}\ ; \tag{12}$$

this is the radius of the circle that fits best to the curve C at the corresponding point.

Without loss of generality, we can assume that (locally) the constraint manifold is given as the graph of a function, $z = \phi(y)$, i.e., the constraint is $g(y, z) = z - \phi(y) = 0$. Write this in the form (11), using y as a parameter ξ, and obtain an expression for the curvature of C at the point $(y, \phi(y))$, in terms of the function ϕ and its derivatives.

(g) Compute the Hessian of the constraint function $g(y, z) = z - \phi(y)$ and write down the term $-\frac{m}{|\nabla g|}(\dot{\mathbf{y}}\cdot \mathrm{Hess}\, g(\mathbf{y})\cdot\dot{\mathbf{y}})$. Use the expression for ρ obtained above to rewrite this term in terms of the centrifugal force,

$$\frac{m\,|\text{velocity}|^2}{\text{radius of curvature}}\ .$$

Discuss your findings.