MATH 3413 Homework 5 Due Thu, Oct 5

Sec. 3.5:

- problems 5, 10 – in these two problems you have to write down the general solution of the non-homogeneous equation; you have to find the exact values of the constants in the particular solution $y_p(x)$ of the non-homogeneous equation; in problem 5 you will first have to apply some trigonometry to transform the right-hand side into a form you know how to deal with;
- 26 – do *not* find the values of the constants in $y_p(x)$, just write down in what form you will be looking for it;
- problems 34, 43(a).

Sec. 3.6: problem 1 – first you have to find the general solution of this problem, then apply the initial conditions to find the values of the constants C_1 and C_2, and then plot the solution $x(t)$ of the initial value problem, and on the plot indicate the period of $x(t)$. To plot the functions, say $\sin x$ and $\frac{1}{4}x^2$ for $-3 \leq x \leq 3$, in Mathematica, type

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Plot[{Sin[x], x^2/4}, {x, -3, 3}]
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then press the SHIFT key and, while holding it down, press RETURN.

Additional problem 1.

(a) Write the second-order linear ODE $x'' + 25x = 0$ as a first-order system. Rewrite this system in the form

$$\mathbf{X}'(t) = \mathsf{A} \cdot \mathbf{X}(t) \ ,$$

where $\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is the unknown vector-function, $\mathbf{X}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}$ is its derivative, $\mathsf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is the matrix of the coefficients of the first-order system, and the dot stands for the matrix-vector multiplication defined by

$$\mathsf{A} \cdot \mathbf{X}(t) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} a_{11}x_1(t) + a_{12}x_2(t) \\ a_{21}x_1(t) + a_{22}x_2(t) \end{pmatrix} \ .$$

Solve the ODE and then write its solution as a solution of the first-order system. Check that it is indeed a solution. Prove that the point $(x_1(t), x_2(t))$ moves on an ellipse in the (x_1, x_2)-plane as t changes. Find the semi-axes of this ellipse. Sketch by hand this ellipse in the (x_1, x_2)-plane.

(b) Write the second-order linear ODE $x'' + 6x' + 25x = 0$ as a first-order system. Rewrite this system in the form

$$\mathbf{X}'(t) = \mathsf{A} \cdot \mathbf{X}(t) \ .$$

Solve the ODE and then write its solution as a solution of the first-order system. Sketch by hand the solution of the second-order system as a curve in the (x_1, x_2)-plane. How does the point $(x_1(t), x_2(t))$ behave as $t \to \infty$?

Additional problem 2.

(a) Find the general solution of the equation

$$x'' + 6x' + 25x = f_0 \, \cos \omega t \tag{1}$$

describing a damped oscillator driven by a periodic external force of amplitude $f_0 =$ const and (angular) frequency ω. (Find the values of the constants in $x_p(t)$, which are going to depend on ω.) Identify the *transient* and the *persistent* parts of the general solution.

(b) Show that, for any choice of constants α and β and of frequency ω, it is true that

$$\alpha \cos(\omega t) + \beta \sin(\omega t) = \gamma \cos(\omega t - \phi_0) \tag{2}$$

for some γ and ϕ_0. Write γ and ϕ_0 in terms of α and β, and then, conversely, write α and β as functions of γ and ϕ_0. This identity shows that each linear combination (with constant coefficients) of $\cos(\omega t)$ and $\sin(\omega t)$ is again a harmonic motion of frequency ω. The constants γ and ϕ_0 are often referred to as the *amplitude* of the motion and the *phase difference* between the external forcing and the motion of the system.

(c) Using (2), rewrite the persistent part $x_p(t)$ of the solution found in (a) in the form $x_p(t) = \gamma \cos(\omega t + \phi_0)$. Write the amplitude γ as a function of the external forcing frequency ω. Using some software, plot γ versus ω for $0 \leq \omega \leq 20$; assume that $f_0 = 1$. Find the value of ω for which the amplitude γ reaches its maximum value.

Hint: You have to show that the function $\gamma(\omega)$ giving the amplitude as a function of the external forcing frequency is

$$\gamma(\omega) = \frac{|f_0|}{\sqrt{(25 - \omega^2)^2 + (6\omega)^2}} \ .$$

You may use (without proving it) the fact that

$$\gamma'(\omega) = \frac{2\omega(7 - \omega^2)\,|f_0|}{[(25 - \omega^2)^2 + (6\omega)^2]^{3/2}} \ .$$