

**Sec. 3.5:**

- problems 5, 10 – in these two problems you have to write down the general solution of the non-homogeneous equation; you have to find the exact values of the constants in the particular solution  $y_p(x)$  of the non-homogeneous equation; in problem 5 you will first have to apply some trigonometry to transform the right-hand side into a form you know how to deal with;
- 26 – do *not* find the values of the constants in  $y_p(x)$ , just write down in what form you will be looking for it;
- problems 34, 43(a).

**Sec. 3.6:** problem 1 – first you have to find the general solution of this problem, then apply the initial conditions to find the values of the constants  $C_1$  and  $C_2$ , and then plot the solution  $x(t)$  of the initial value problem, and on the plot indicate the period of  $x(t)$ . To plot the functions, say  $\sin x$  and  $\frac{1}{4}x^2$  for  $-3 \leq x \leq 3$ , in Mathematica, type

```
Plot[{Sin[x], x^2/4}, {x, -3, 3}]
```

then press the SHIFT key and, while holding it down, press RETURN.

**Additional problem 1.**

- (a) Write the second-order linear ODE  $x'' + 25x = 0$  as a first-order system. Rewrite this system in the form

$$\mathbf{X}'(t) = \mathbf{A} \cdot \mathbf{X}(t) ,$$

where  $\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$  is the unknown vector-function,  $\mathbf{X}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}$  is its derivative,  $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$  is the matrix of the coefficients of the first-order system, and the dot stands for the matrix-vector multiplication defined by

$$\mathbf{A} \cdot \mathbf{X}(t) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} a_{11}x_1(t) + a_{12}x_2(t) \\ a_{21}x_1(t) + a_{22}x_2(t) \end{pmatrix} .$$

Solve the ODE and then write its solution as a solution of the first-order system. Check that it is indeed a solution. Prove that the point  $(x_1(t), x_2(t))$  moves on an ellipse in the  $(x_1, x_2)$ -plane as  $t$  changes. Find the semi-axes of this ellipse. Sketch by hand this ellipse in the  $(x_1, x_2)$ -plane.

- (b) Write the second-order linear ODE  $x'' + 6x' + 25x = 0$  as a first-order system. Rewrite this system in the form

$$\mathbf{X}'(t) = \mathbf{A} \cdot \mathbf{X}(t) .$$

Solve the ODE and then write its solution as a solution of the first-order system. Sketch by hand the solution of the second-order system as a curve in the  $(x_1, x_2)$ -plane. How does the point  $(x_1(t), x_2(t))$  behave as  $t \rightarrow \infty$ ?

### Additional problem 2.

- (a) Find the general solution of the equation

$$x'' + 6x' + 25x = f_0 \cos \omega t \quad (1)$$

describing a damped oscillator driven by a periodic external force of amplitude  $f_0 = \text{const}$  and (angular) frequency  $\omega$ . (Find the values of the constants in  $x_p(t)$ , which are going to depend on  $\omega$ .) Identify the *transient* and the *persistent* parts of the general solution.

- (b) Show that, for any choice of constants  $\alpha$  and  $\beta$  and of frequency  $\omega$ , it is true that

$$\alpha \cos(\omega t) + \beta \sin(\omega t) = \gamma \cos(\omega t - \phi_0) \quad (2)$$

for some  $\gamma$  and  $\phi_0$ . Write  $\gamma$  and  $\phi_0$  in terms of  $\alpha$  and  $\beta$ , and then, conversely, write  $\alpha$  and  $\beta$  as functions of  $\gamma$  and  $\phi_0$ . This identity shows that each linear combination (with constant coefficients) of  $\cos(\omega t)$  and  $\sin(\omega t)$  is again a harmonic motion of frequency  $\omega$ . The constants  $\gamma$  and  $\phi_0$  are often referred to as the *amplitude* of the motion and the *phase difference* between the external forcing and the motion of the system.

- (c) Using (2), rewrite the persistent part  $x_p(t)$  of the solution found in (a) in the form  $x_p(t) = \gamma \cos(\omega t + \phi_0)$ . Write the amplitude  $\gamma$  as a function of the external forcing frequency  $\omega$ . Using some software, plot  $\gamma$  versus  $\omega$  for  $0 \leq \omega \leq 20$ ; assume that  $f_0 = 1$ . Find the value of  $\omega$  for which the amplitude  $\gamma$  reaches its maximum value.

*Hint:* You have to show that the function  $\gamma(\omega)$  giving the amplitude as a function of the external forcing frequency is

$$\gamma(\omega) = \frac{|f_0|}{\sqrt{(25 - \omega^2)^2 + (6\omega)^2}} .$$

You may use (without proving it) the fact that

$$\gamma'(\omega) = \frac{2\omega(7 - \omega^2) |f_0|}{[(25 - \omega^2)^2 + (6\omega)^2]^{3/2}} .$$