MATH 4073 Homework 5 Due Fri, 10/14/16

Problem 1. [Using Taylor expansions to find approximate solutions of equations]

In this problem you will find approximate solutions of the nonlinear equation

$$x + \mathrm{e}^{0.00001x} = 100 \ . \tag{1}$$

(a) You can give a *very rough estimate* of the solution thinking like this. Clearly, the left-hand side of equation (1) is a strictly increasing function of x (look at its derivative). If $x = 100$, then the left-hand side is has value $100 + \mathrm{e}^{0.00001\cdot 100} = 100 + \mathrm{e}^{0.01}$, which is a little more than 101, so that the root we are looking for must be a little less than 100. Since x in $\mathrm{e}^{0.00001x}$ is multiplied by the very small number 0.00001, for $x \approx 100$, we will have $\mathrm{e}^{0.00001x} \approx \mathrm{e}^{0.001} \approx 1$. Use the Taylor expansion of e^z about $z = 0$, truncated right after the constant term, i.e.,

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \approx 1 \ ,$$

in order to find an approximate value of the solution x of equation (1).

(b) Now follow the ideas of part (a) and use Taylor expansion of the exponent truncated right after the linear term,

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \approx 1 + z \ ,$$

to obtain an approximation of the solution that is better than the one obtained in part (a).

(c) Run the code `newton.m` (available at the class web-site) to find the exact root of the nonlinear equation (1) (set the tolerance to be small, say, 10^{-14}); attach your MATLAB printout.

(d) Find the absolute and the relative errors of the approximate solutions found in parts (a) and (b).

Problem 2. [Error bounds in piecewise-linear Lagrange interpolation]

In this problem you will study in detail the piecewise-linear interpolation of the function

$$f(x) = \frac{1}{x} \tag{2}$$

on the interval $[1, 2]$, and then on the interval $[1, 3]$. The graphs of the function and the Lagrange interpolating polynomial on the interval $[1, 2]$ are shown in Figure 1.

(a) Find the first order Lagrange polynomial $P_1(x)$ of $f(x) = \frac{1}{x}$ that passes through the points $(1, f(1))$ and $(2, f(2))$.

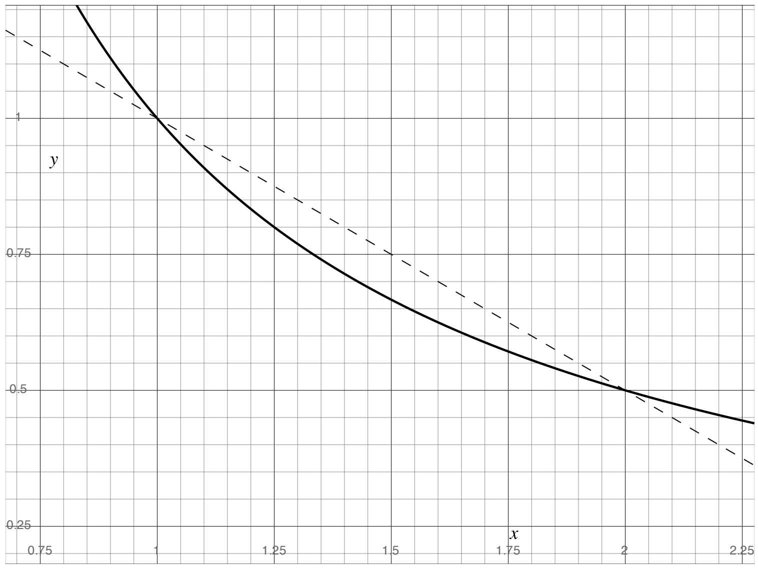

Figure 1: Linear interpolation of $f(x) = \frac{1}{x}$ on the interval $[1, 2]$ (the function f is plotted with a thick solid line, and the interpolating polynomial with a dashed line).

(b) Let

$$E := \max_{x\in[1,2]} |f(x) - P_1(x)|$$

be the *true* error of the first order Lagrange interpolation. Find the numerical value of E.

Hint: You first have to find the value x^* of the argument that maximizes the expression $|f(x) - P_1(x)|$. Note that f is concave up, so that the graph of P_1 lies above the graph of f, therefore $|f(x) - P_1(x)| = P_1(x) - f(x)$.

(c) Find the rigorous error bound of the linear interpolation on $[1, 2]$ given by Theorem 3.3 in Section 3.1 of the book:

$$|f(x) - P_n(x)| \le \frac{1}{(n+1)!} \max_{x\in[1,2]} \left(\left| f^{(n+1)}(\xi(x)) \right| \prod_{j=0}^{n} |x - x_j| \right) . \tag{3}$$

Since that you do not know the value of $\xi(x)$ in this bound, the only thing you can do to end up with a rigorous upper bound is to take maximum over $\xi \in [1, 2]$ and over $x \in [1, 2]$ separately, and use the obvious inequality

$$\max_{x\in[1,2]} \left(\left| f^{(n+1)}(\xi(x)) \right| \prod_{j=0}^{n} |x - x_j| \right) \le \max_{\xi\in[1,2]} \left| f^{(n+1)}(\xi) \right| \max_{x\in[1,2]} \prod_{j=0}^{n} |x - x_j|$$

(food for thought: why is this obvious?). Computing the expressions in the right-hand side is quite easy in the case $n = 1$ which you are considering; note that, for $x \in [x_0, x_1] := [1, 2]$,

$$\prod_{j=0}^{1} |x - x_j| = |x - 1|\,|x - 2| = (x - 1)(2 - x) .$$

Find the exact value of this bound (i.e., of the expression in the right-hand side of (3)), and compute its numerical value. Compare with the exact value of the error found in part (b); discuss briefly.

(d) Now find the Lagrange interpolating polynomial of f over the interval $[2, 3]$, and write your results from parts (a) and (c) together in the form

$$P_{\text{piece-lin}}(x) = \begin{cases} b_1 x + c_1 \ , & x \in [1, 2] \ , \\ b_2 x + c_2 \ , & x \in [2, 3] \ . \end{cases}$$

(e) Use your result from part (d) to compute $P_{\text{piece-lin}}(1.25)$, and compare its value with $f(1.25)$.

(f) Finally, compute the Taylor series of f around $x_0 = 1$. Does it converge for $x = 2$?

Hint: Note that $\dfrac{1}{x} = \dfrac{1}{1 + (x-1)} = \dfrac{1}{1 - [-(x-1)]}$, and use the formula for the sum of a geometric series; for which values of $|x - 1|$ does this series converge?

Problem 3. [Quadratic Lagrange interpolation]

This problem is a continuation of Problem 2.

(a) Construct the Lagrange interpolating polynomial of degree at most 2, $P_2(x)$, to the function $f(x)$ given by (2) in the interval $[1, 3]$. The polynomial $P_2(x)$ is the only quadratic function whose graph goes through the points $(1, f(1)) = (1, 1)$, $(2, f(2)) = (2, \frac{1}{2})$ and $(3, f(3)) = (3, \frac{1}{3})$.

(b) Use the quadratic Lagrange interpolating polynomial found in part (a) to compute the approximate value of $P_2(1.25)$. Find the numerical value of the absolute error $|f(1.25) - P_2(1.25)|$.

Problem 4. [Using Lagrange interpolants to find approximate value of integrals]

This problem is a continuation of Problems 2 and 3. Let

$$I_{\text{exact}} := \int_1^3 \frac{1}{x}\, dx \ , \quad I_{\text{piece-lin}} := \int_1^3 P_{\text{piece-lin}}(x)\, dx \ , \quad I_{\text{quadr}} := \int_1^3 P_2(x)\, dx$$

be the definite integrals from 1 to 3 of the function $f(x) = \frac{1}{x}$ given by (2), and the piecewise-linear and the quadratic interpolating functions you computed in Problems 2 and 3.

(a) *Without computing anything*, decide which of the numbers I_{exact} and $I_{\text{piece-lin}}$ is larger. A (hand-drawn) picture and a couple of sentence of explanation are enough.

(b) Compute the numerical values of I_{exact}, $I_{\text{piece-lin}}$, and I_{quadr}. Was your prediction in part (a) correct?

(c) Compute the numerical values of the absolute errors in approximating I_{exact} by $I_{\text{piece-lin}}$ and by I_{quadr}.

Problem 5. [Newton's divided differences interpolating polynomial]

The purpose of this problem is to construct and study the Newton's divided difference form of the interpolating polynomial,

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, \dots, x_k] \prod_{j=0}^{k-1} (x - x_j) \ ,$$

to the function $f(x) = \cos(\pi x)$. The points x_i, $i = 0, 1, 2, 3$ used to construct the interpolating polynomial are given in the table below. Figure 2 shows the graphs of the function $f(x) = \cos(\pi x)$ and the interpolating polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$, and $P_3(x)$.

(a) Compute the missing entries in the divided differences table below. Write your calculations clearly and leave the coefficients in symbolic form (i.e., do not compute the *numerical values* of things like $12(8\sqrt{2} - 11)$).

x_i	0^{th} order	1^{st} order	2^{nd} order	3^{rd} order
$x_0 = 0$	$f[x_0] = 1$			
		$f[x_0, x_1] =?$		
$x_1 = \frac{1}{3}$	$f[x_1] = \frac{1}{2}$		$f[x_0, x_1, x_2] =?$	
		$f[x_1, x_2] =?$		$f[x_0, x_1, x_2, x_3] = 12(8\sqrt{2} - 11)$
$x_2 = \frac{1}{2}$	$f[x_2] = 0$		$f[x_1, x_2, x_3] = 12(2\sqrt{2} - 3)$	
		$f[x_2, x_3] = -2\sqrt{2}$		
$x_3 = \frac{1}{4}$	$f[x_3] =?$			

(b) Write down the interpolating polynomial $P_0(x)$ based on the values in the divided differences table above. ($P_0(x)$ should "agree" with $f(x)$ at the point x_0.)

(c) Similarly to part (b), write down the interpolating polynomial $P_1(x)$ based on the values in the divided differences table above. ($P_1(x)$ should "agree" with $f(x)$ at the points x_0 and x_1.)

(d) Similarly to part (b), write down the interpolating polynomial $P_2(x)$ based on the values in the divided differences table above. Do *not* expand it – just substitute the coefficients in the Newton's divided difference interpolating polynomial with the corresponding entries from the table. ($P_2(x)$ should "agree" with $f(x)$ at the points x_0, x_1, and x_2.)

(e) Similarly to part (b), write down the interpolating polynomial $P_3(x)$ based on the values in the divided differences table above. Do *not* expand the polynomial!

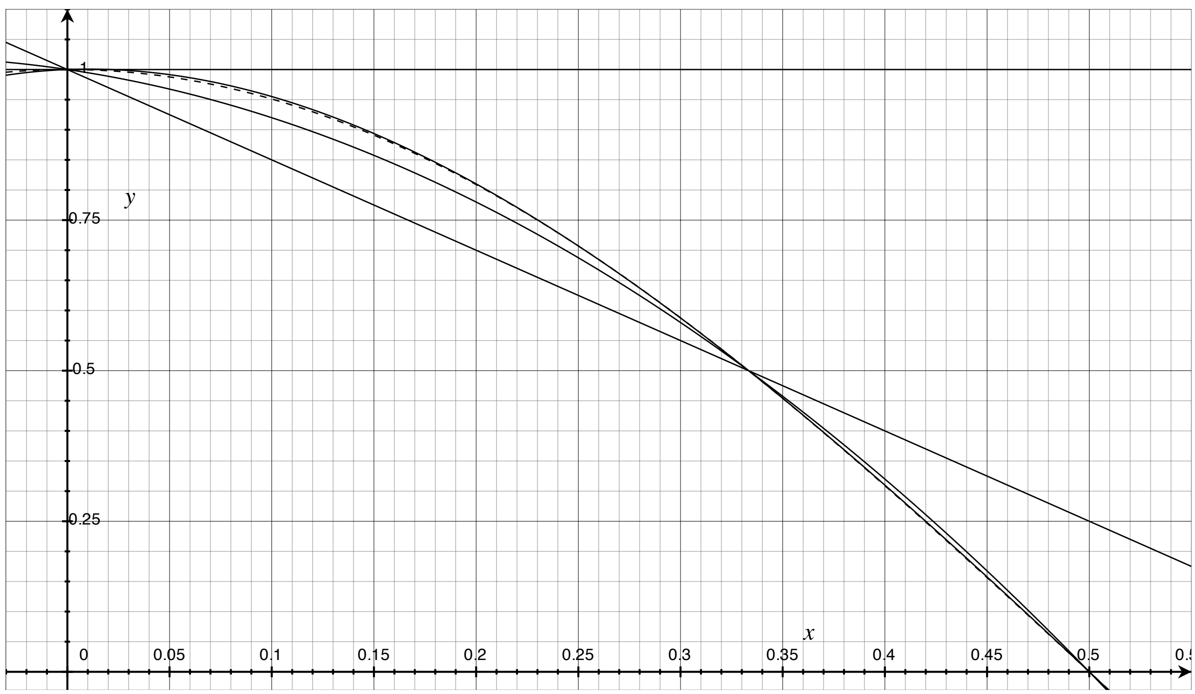

Figure 2: Graphs of the function $f(x)$ (the dashed line) and the interpolating polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$, and $P_3(x)$.