MATH 4163 Homework 4 Due at noon on Fri, Sep 28, 2012

Problem 1. The wave equation,

$$u_{tt}(\mathbf{r},t) = c^2 \Delta u(\mathbf{r},t) \ , \qquad \mathbf{r} = (x_1, \ldots, x_d) \in \mathbb{R}^d \ ,$$

describes the propagation of waves in the d-dimensional space. Here $\Delta := \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ is the Laplace's operator (Laplacian), and $c > 0$ is a constant (which is the speed of the wave).

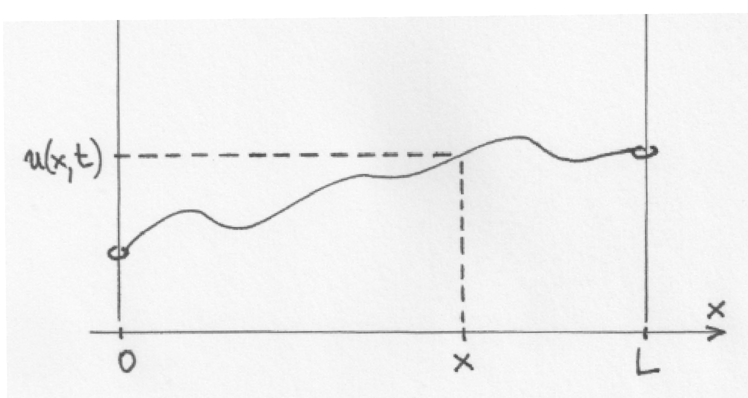
In the case of one spatial dimension (i.e., for $d = 1$), the wave equation describes the motion of a spring vibrating in a fixed plane, i.e., at time t, the position of the string is described by the equation $y = u(x,t)$. The derivation of the wave equation for $d = 1$ is given in Section 4.2 of the book, and the physical meaning of the BCs is discussed in Section 4.3.

In this problem you will solve the following BVP for the wave equation with homogeneous Neumann BCs:

$$\begin{aligned} & u_{tt}(x,t) = u_{xx}(x,t) \ , \qquad x \in [0,L] \ , \quad t \in [0,\infty) \ , \\ & u_x(0,t) = 0 \ , \quad u_x(L,t) = 0 \ , \\ & u(x,0) = f(x) \ , \quad u_t(x,0) = g(x) \ , \end{aligned} \tag{1}$$

where you can assume that you know the Fourier cosine or Fourier sine series (whichever one you need) of the functions f and g giving the initial position and the initial velocity of the points of the string.

The meaning of the BCs is the following: the ends of the string are attached to small rings (with negligible mass) that can slide without friction on two thin vertical rods, as shown in the figure below (which gives the position of the points of the string at time t).



(a) Set $u(x,t) = X(x)\,T(t)$ and obtain the equations for T and X coming from the separation of variables.

(b) Solve the BVP for X coming from the separation of variables and the BCs.

Hint: You have done this in previous homework problems. You should obtain a set of functions X_n where $n = 0, 1, 2, \ldots$.

(c) Write down the equations for the functions T_n and find their general solutions. You have to consider the cases $n = 0$ and $n \in \mathbb{N}$ separately. For each n, the function T_n must have two arbitrary constants, say A_n and B_n (because the equations for T_n are of second order).

(d) Write the solution $u(x,t)$ of the BVP (1) as a superposition of the functions $u_n(x,t) = X_n(x)\,T_n(t)$, and apply the ICs to determine the constants A_n and B_n.

(e) The general theory says that the each solution of the wave equation in one spatial dimension can be written in the form

$$u(x,t) = \phi(x - ct) + \psi(x + ct) \ , \tag{2}$$

where ϕ and ψ are functions of one variable. The terms $\phi(x-ct)$ and $\psi(x+ct)$ describe waves moving to the right, resp. to the left, with speed c. Write the solution of the BVP (1) found in part (d) in the form (2).

Hint: You will need the trigonometric identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \ , \quad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \ .$$

(f) The expression $y_c(t) = \dfrac{1}{L}\displaystyle\int_0^L u(x,t)\,\mathrm{d}x$ gives the y-coordinate of the center of mass of the string at time t. Find $y_c(t)$ for the solution you obtained in part (d). The computation is very simple because $\displaystyle\int_0^L \cos\frac{n\pi x}{L}\,\mathrm{d}x = 0$, so that most terms in $u(x,t)$ will not contribute to $y_c(t)$. Look at the ICs and give a physical interpretation of your result for $y_c(t)$. Use this to give a physical interpretation of the Fourier sine/cosine coefficient(s) of f and g that occur in the expression for $y_c(t)$.

Problem 2. Consider the following Sturm-Liouville eigenvalue problem:

$$\begin{aligned} &\phi''(x) + \lambda\phi(x) = 0 \ , \qquad x \in [0, L] \ , \\ &\phi(0) = 0 \ , \quad \phi'(L) = 0 \ . \end{aligned} \tag{3}$$

(a) Find the eigenvalues λ_n and the corresponding eigenfunctions ϕ_n of the Sturm-Liouville eigenvalue problem (3).

(b) From the expressions for λ_n show that there is a smallest one, and that $\lim\limits_{n\to\infty} \lambda_n = \infty$.

(c) From the explicit expressions for the eigenfunctions ϕ_n show that ϕ_n has exactly $(n-1)$ zeros.

(d) Use the explicit expressions for the eigenfunctions ϕ_n to show that the eigenfunctions ϕ_n and ϕ_k are orthogonal if $n \neq k$, with respect to an appropriately defined inner product. What is the inner product that you used, and why? (The answer to the last question comes from the general theory.)

Hint: Use the trigonometric identity (following from the hint to Problem 1(e))

$$\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right] .$$

Problem 3. Solve the BVP for the heat equation in one spatial dimension with a homogeneous Dirichlet BC at the left end and a homogeneous Robin condition at the right end:

$$\begin{aligned} &u_t(x,t) = \alpha^2 u_{xx}(x,t) , \qquad x \in [0,1] , \quad t \in [0,\infty) , \\ &u(0,t) = 0 , \quad u_x(1,t) = -hu(1,t) , \\ &u(x,0) = f(x) , \end{aligned} \tag{4}$$

where α and h are positive constants, and f is a piecewise smooth function. Let the numbers z_n (with $n \in \mathbb{N}$) be the solutions of the transcendental equation

$$\tan z = -\frac{z}{h} ,$$

in increasing order: $z_1 < z_2 < z_3 < \ldots$.

(a) Write the BVP for X_n coming from the separation of variables, and express its eigenvalues and eigenfunctions in terms of the numbers z_n.

(b) Find the functions T_n and write down the solution $u(x,t)$ of (4) as a superposition of the functions $u_n(x,t) = X(x)\,T(t)$; do not impose the ICs.

(c) Use the IC to find the coefficients in the expansion of $u(x,t)$ from part (b).

Hint: This is easy if you apply the general theory.

Problem 4. Consider the Sturm-Liouville eigenvalue problem with a Neumann BC at the left end and a Robin BC at the right end (with $h > 0$):

$$\begin{aligned} &\phi''(x) + \lambda\phi(x) = 0 , \qquad x \in [0,L] , \\ &\phi'(0) = 0 , \quad \phi'(L) = -h\phi(L) . \end{aligned}$$

(a) Derive the transcendental equation that determines the eigenvalues λ_n, and draw a sketch similar to Figure 5.8.1 in the book that illustrates graphically the behavior of the roots of that equation (consider only the case $h > 0$).

(b) Write down the eigenfunction ϕ_n corresponding to the eigenvalue λ_n.

(c) Write down the orthogonality relation for the eigenfunctions that follows from the general theory.