MATH 4443 **Homework 5** **Due 2/22/18 (Thursday)**

Problem 1. Assume that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$.

(a) Show that the function $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ is defined on $(-R, R)$.

(b) Show that the function F satisfies $F'(x) = f(x)$.

(c) Antiderivatives are not unique. If g is an arbitrary function satisfying $g'(x) = f(x)$ on $(-R, R)$, find a power series representation for g.

Problem 2.

(a) If the number s satisfies $0 < s < 1$, show that ns^{n-1} is bounded for all $n \geq 1$.

Hint: Look at the ratio of two consecutive terms, $\frac{(n+1)s^n}{ns^{n-1}}$ – how does it behave for very large values of n?

(b) Given an arbitrary $x \in (-R, R)$, pick t to satisfy $|x| < t < R$. Prove that the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges, which will provide a proof of Theorem 6.5.6 of Abbott's book.

Hint: One way to show the convergence is to note that

$$\sum_{n=1}^{\infty} |n a_n x^{n-1}| = \sum_{n=1}^{\infty} \frac{1}{t} \left(n \left| \frac{x}{t} \right|^{n-1} \right) |a_n t^n| \ ,$$

and to use the result from part (a) with $s = \left| \frac{x}{t} \right|$.

Problem 3. A series $\sum_{n=0}^{\infty} a_n$ is called *Abel summable to* L if the power series $f(x) := \sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in [0, 1)$ and $L = \lim_{x \to 1^-} f(x)$.

(a) Show that any series that converges to a limit L is also Abel summable. Please give a clear and detailed argument.

Hint: Assume that $\sum a_n$ converges to L. What does this imply about the convergence of the power series $f(x) = \sum a_n x^n$ at $x = 1$? What does Abel's Theorem imply about the convergence of the power series on $[0, 1]$? What can you conclude about the properties of f on $[0, 1]$? What does this imply about the limit of $f(x)$ as $x \to 1^-$?

(b) Show that the series $\sum_{n=0}^{\infty}(-1)^n$ is Abel summable and find its sum.

Hint: The formula for the geometric series will be useful.

Problem 4.

(a) The derivation of the Taylor series for $\arctan x$ is valid for all $x \in (-1, 1)$. Notice, however, that the series also converges for $x = 1$. What does Abel's Theorem imply about the convergence of the power series over the interval $[0, 1]$?

Remark: Note that Theorem 6.5.2 will not be enough in this case.

(b) What can you conclude about the continuity of arctan on $[0, 1]$? Which theorem from the book helps you come to this conclusion?

(c) Use your result from part (b) to explain why the value of the series for $\arctan x$ at $x = 1$ must necessarily be $\arctan 1$.

(d) What identity do you get for $x = 1$? (It is sometimes called the *Leibniz's identity.*)

Problem 5. Recall the Taylor series of $\cos y$ for $y \in \mathbb{R}$, $\frac{1}{1+y^2}$ for $y \in (-1, 1)$, and $\ln(1 + y)$ for $y \in (-1, 1]$. The series of $\ln(1 + y)$ is obtained by antidifferentiating

$$\frac{1}{1+y} = \frac{1}{1-(-y)} = 1 - y + y^2 - y^3 + y^4 - \cdots \ .$$

Manipulate these series to obtain Taylor series representations for each of the following functions. In each case, write down the interval in which the series converges.

(a) $f(x) = x\cos(x^2)$ (b) $g(x) = \dfrac{x}{(1+4x^2)^2}$ (c) $h(x) = \ln(1 + x^2)$

Problem 6. In this problem you will demonstrate that, if we take the power series representation of the exponential function to be its definition, then familiar statements like $(\exp x)' = \exp x$ and $\exp(-x) = (\exp x)^{-1}$ follow naturally.

Define the function $\exp x$ by its Taylor series representation, $\exp x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$

(a) Prove that the Taylor series of $\exp x$ converges uniformly on any interval $[-R, R]$.

(b) Use the Term-by-term Differentiability Theorem to find the derivative of $\exp x$.

(c) Recall that the *Cauchy product* of two series, $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is defined as the series $\sum_{n=0}^{\infty} c_n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Find the Cauchy product of $\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\exp(-x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$. You will need to use Newton's binomial formula,

$$(1+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k \ ,$$

where $\binom{n}{k} := \frac{n!}{k!\,(n-k)!}$ are the binomial coefficients $(0! := 1)$ to show that all coefficients in the Cauchy product of the series for $\exp x$ and $\exp(-x)$ but the constant one are zero.

Food for Thought: Abbott, Exercises 6.5.1, 6.5.6, 6.6.7.

Hint for Abbott, Exercise 6.6.7:

(a) One example would be the function $g(x) = \frac{1}{1+x^2}$ considered in class.

(b) Let $h(x) = \sin x + g(x)$, where $g(x)$ is the "Counterexample" function from page 203.

(c) Let

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ e^{-1/x^2} & \text{for } x > 0\ . \end{cases}$$