Problem 1. Assume that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$.

(a) Show that the function $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ is defined on $(-R, R)$.

(b) Show that the function $F$ satisfies $F'(x) = f(x)$.

(c) Antiderivatives are not unique. If $g$ is an arbitrary function satisfying $g'(x) = f(x)$ on $(-R, R)$, find a power series representation for $g$.

Problem 2.

(a) If the number $s$ satisfies $0 < s < 1$, show that $ns^{n-1}$ is bounded for all $n \geq 1$.

Hint: Look at the ratio of two consecutive terms, $\frac{(n+1)s^n}{ns^{n-1}}$ — how does it behave for very large values of $n$?

(b) Given an arbitrary $x \in (-R, R)$, pick $t$ to satisfy $|x| < t < R$. Prove that the series $\sum_{n=1}^{\infty} na_n x^{n-1}$ converges, which will provide a proof of Theorem 6.5.6 of Abbott’s book.

Hint: One way to show the convergence is to note that

$$\sum_{n=1}^{\infty} |na_n x^{n-1}| = \sum_{n=1}^{\infty} \frac{1}{t} \left( n \left| \frac{x}{t} \right|^{n-1} \right) |a_n t^n|,$$

and to use the result from part (a) with $s = \left| \frac{x}{t} \right|$.

Problem 3. A series $\sum_{n=0}^{\infty} a_n$ is called Abel summable to $L$ if the power series $f(x) := \sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in [0, 1)$ and $L = \lim_{x \to 1^-} f(x)$.

(a) Show that any series that converges to a limit $L$ is also Abel summable. Please give a clear and detailed argument.

Hint: Assume that $\sum_{n} a_n$ converges to $L$. What does this imply about the convergence of the power series $f(x) = \sum_{n} a_n x^n$ at $x = 1$? What does Abel’s Theorem imply about the convergence of the power series on $[0, 1]$? What can you conclude about the properties of $f$ on $[0, 1]$? What does this imply about the limit of $f(x)$ as $x \to 1^-$?
(b) Show that the series \( \sum_{n=0}^{\infty} (-1)^n \) is Abel summable and find its sum.

*Hint:* The formula for the geometric series will be useful.

**Problem 4.**

(a) The derivation of the Taylor series for \( \arctan x \) is valid for all \( x \in (-1, 1) \). Notice, however, that the series also converges for \( x = 1 \). What does Abel’s Theorem imply about the convergence of the power series over the interval \([0, 1]\)?

*Remark:* Note that Theorem 6.5.2 will not be enough in this case.

(b) What can you conclude about the continuity of \( \arctan \) on \([0, 1]\)? Which theorem from the book helps you come to this conclusion?

(c) Use your result from part (b) to explain why the value of the series for \( \arctan x \) at \( x = 1 \) must necessarily be \( \arctan 1 \).

(d) What identity do you get for \( x = 1 \)? (It is sometimes called the *Leibniz’s identity*.)

**Problem 5.** Recall the Taylor series of \( \cos y \) for \( y \in \mathbb{R} \), \( \frac{1}{1+y^2} \) for \( y \in (-1, 1) \), and \( \ln(1 + y) \) for \( y \in (-1, 1] \). The series of \( \ln(1 + y) \) is obtained by antidifferentiating

\[
\frac{1}{1+y} = \frac{1}{1-(-y)} = 1 - y + y^2 - y^3 + y^4 - \cdots.
\]

Manipulate these series to obtain Taylor series representations for each of the following functions. In each case, write down the interval in which the series converges.

(a) \( f(x) = x \cos(x^2) \)
(b) \( g(x) = \frac{x}{(1 + 4x^2)^2} \)
(c) \( h(x) = \ln(1 + x^2) \)

**Problem 6.** In this problem you will demonstrate that, if we take the power series representation of the exponential function to be its definition, then familiar statements like \( (\exp x)' = \exp x \) and \( \exp(-x) = (\exp x)^{-1} \) follow naturally.

Define the function \( \exp x \) by its Taylor series representation, \( \exp x := \sum_{n=0}^{\infty} \frac{x^n}{n!} \)

(a) Prove that the Taylor series of \( \exp x \) converges uniformly on any interval \([-R, R]\).

(b) Use the Term-by-term Differentiability Theorem to find the derivative of \( \exp x \).
(c) Recall that the Cauchy product of two series, $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is defined as the series $\sum_{n=0}^{\infty} c_n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Find the Cauchy product of $\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\exp(-x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$. You will need to use Newton’s binomial formula,

\[
(1 + y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k,
\]

where $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ are the binomial coefficients ($0! := 1$) to show that all coefficients in the Cauchy product of the series for $\exp x$ and $\exp(-x)$ but the constant one are zero.

**Food for Thought:** Abbott, Exercises 6.5.1, 6.5.6, 6.6.7.

*Hint for Abbott, Exercise 6.6.7:*

(a) One example would be the function $g(x) = \frac{1}{1+x^2}$ considered in class.

(b) Let $h(x) = \sin x + g(x)$, where $g(x)$ is the “Counterexample” function from page 203.

(c) Let

\[
f(x) = \begin{cases} 
0 & \text{for } x \leq 0, \\
e^{-1/x^2} & \text{for } x > 0 .
\end{cases}
\]