MATH 5163 **Homework 5** **Due Fri, 2/21/14, 4 p.m.**

Problem 1. [Fundamental solution of the heat equation on $\mathbb{R}^{1+1}$]
In this problem you will derive the so-called *fundamental solution* of the heat equation on $\mathbb{R}^{1+1}$, i.e., in one spatial dimension (x) and one temporal dimension (t). Write the heat equation as

$$u_t(x,t) = \alpha^2 u_{xx}(x,t) \ , \qquad x \in \mathbb{R} , \quad t \in (0,\infty) \ , \tag{1}$$

where α is a positive constant. We demand that the fundamental solution have the following properties:

(i) $u(x,t) \geq 0$;

(ii) $u(\cdot,t) \in C^\infty(\mathbb{R})$ for all $t > 0$;

(iii) $\displaystyle\int_{\mathbb{R}} u(x,t)\,\mathrm{d}x = 1$ for all $t > 0$;

(iv) $\displaystyle\lim_{x\to\pm\infty} u(t,x) = 0$ for all $t > 0$;

(v) $u(t,x) = u(t,-x)$ for all $t > 0$.

To see that such a solution exists, first make the assumption that $u(x,t) = \dfrac{1}{\alpha\sqrt{t}}\, v(\zeta)$, where $\zeta := \dfrac{x}{\alpha\sqrt{t}}$, and v is a function of one variable, which is (hopefully) defined for all $\zeta \in \mathbb{R}$.

(a) Show that if u satisfies (1), v must satisfy the ODE $\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[v'(\zeta) + \frac{\zeta}{2}\, v(\zeta)\right] = 0$.

(b) Using some of the conditions (i)–(iv) above, argue that v is an even function, that $v'(0) = 0$, and that $\displaystyle\lim_{\zeta\to\pm\infty} v(\zeta) = 0$. Please specify which condition(s) you are using in each case.

(c) Show that v satisfies the ODE $v'(\zeta) + \frac{\zeta}{2}\, v(\zeta) = 0$.

(d) Solve the equation for v derived in part (c); your answer will have the constant $v(0)$ which will be determined later. Write your result for v in terms of the function u.

(e) Use some of the conditions (i)–(iv) above to find $v(0)$, and write the final expression for the fundamental solution u of (1).

(f) In Remark 2 to Problem 5 of Homework 4 it was mentioned that the family of functions $\dfrac{1}{2\sqrt{\pi\varepsilon}}\mathrm{e}^{-x^2/(4\varepsilon)}$ (defined for all $\varepsilon > 0$) converges to δ in $\mathscr{D}'(\mathbb{R})$ as $\varepsilon \to 0^+$. Using this fact, find the limit of the fundamental solution of (1) found in part (e) as $t \to 0^+$, considered as a 1-parameter family of functions in $\mathscr{D}(\mathbb{R})$ (where the variable is $x \in \mathbb{R}$ and the parameter is $t > 0$).

Problem 2. [Continuity of P.v.$\frac{1}{x}$ **on** $\mathscr{D}(\mathbb{R})$**]**

(a) Let F be a linear functional on $\mathscr{D}(\Omega)$. Recall that by definition the linear functional F is *continuous* if $\langle F, \phi_k \rangle \to \langle F, \phi \rangle$ as $k \to \infty$ for every sequence ϕ_k in $\mathscr{D}(\Omega)$ that converges to $\phi \in \mathscr{D}(\Omega)$ as $k \to \infty$. Explain why, in order to prove the continuity of F, it is enough to prove that $\langle F, \phi_k \rangle \to 0$ as $k \to \infty$ for every sequence ϕ_k in $\mathscr{D}(\Omega)$ that converges to 0 in $\mathscr{D}(\Omega)$ as $k \to \infty$.

(b) Prove that the linear functional P.v.$\dfrac{1}{x}$ on $\mathscr{D}(\mathbb{R})$ defined by

$$\left\langle \text{P.v.}\frac{1}{x}, \phi \right\rangle := \text{P.v.} \int \frac{\phi(x)}{x}\,\mathrm{d}x\ , \qquad \phi \in \mathscr{D}(\mathbb{R})$$

is continuous.

Hint: Let $\phi_k \to 0$ in $\mathscr{D}(\mathbb{R})$. Write in detail what this means. Then start the proof of the continuity of P.v.$\dfrac{1}{x}$ as follows: let $\phi_k \in \mathscr{D}(\mathbb{R})$ be a sequence converging to 0 in $\mathscr{D}(\mathbb{R})$. Then, for any $R > 0$ large enough so that the all ϕ_k are supported on $[-R, R]$,

$$\left|\left\langle \text{P.v.}\frac{1}{x}, \phi_k \right\rangle\right| = \left|\text{P.v.} \int \frac{\phi_k(x)}{x}\,\mathrm{d}x\right| \overset{*}{=} \left|\text{P.v.} \int_{-R}^{R} \frac{\phi_k(0) + x\phi_k'(y)}{x}\,\mathrm{d}x\right| \le \cdots .$$

What was used is the equality marked with $*$? Continue the above chain of (in)equalities to prove the continuity of P.v.$\dfrac{1}{x}$ on $\mathscr{D}(\mathbb{R})$, clearly indicating at each step what property you have used.

Problem 3. [Derivatives of a delta function]
Prove that $x^m \delta^{(k)} = 0$ for $0 \le k < m$.

Problem 4. $[\frac{\mathrm{d}}{\mathrm{d}x} \ln|x| = \text{P.v.}\frac{1}{x}$ **in** $\mathscr{D}'(\mathbb{R})]$
Let the function $u : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined as $u(x) := \ln|x|$.

(a) Show that $u \in L^1_{\text{loc}}(\mathbb{R})$.

Hint: It is enough to show that $\displaystyle\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 |\ln x|\,\mathrm{d}x < \infty$ and that $\displaystyle\int_1^R \ln x\,\mathrm{d}x < \infty$ for every $R > 1$.

(b) Prove that $u' = \text{P.v.}\frac{1}{x}$ in $\mathscr{D}'(\mathbb{R})$.

Hint: For any $\phi \in \mathscr{D}(\mathbb{R})$ write

$$\langle u', \phi \rangle = -\langle u, \phi' \rangle = -\int \ln|x|\,\phi'(x)\,\mathrm{d}x = -\lim_{\varepsilon \to 0^+} \int_{\{|x| > \varepsilon\}} \ln|x|\,\phi'(x)\,\mathrm{d}x\ ,$$

and integrate by parts.

Problem 5. [$\Delta \ln|\mathbf{x}| = 2\pi\delta(\mathbf{x})$ **in** $\mathscr{D}'(\mathbb{R}^2)$]
In this problem you will show that

$$\Delta \ln|\mathbf{x}| = 2\pi\delta(\mathbf{x}) \ , \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \tag{2}$$

in $\mathscr{D}'(\mathbb{R}^2)$; here $|\mathbf{x}| := \sqrt{x_1^2 + x_2^2}$. It is easy to show that the function

$$u : \mathbb{R}^2 \smallsetminus \{\mathbf{0}\} \to \mathbb{R} : \mathbf{x} \mapsto \ln|\mathbf{x}| \tag{3}$$

is $C^\infty(\mathbb{R}^2 \smallsetminus \{\mathbf{0}\}$ (you do not need to show this here).

(a) Prove that, if you think of u defined in (3) as a C^∞ function on $\mathbb{R}^2 \smallsetminus \{\mathbf{0}\}$, then

$$\Delta u(\mathbf{x}) = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \ln r\right) \ , \qquad \mathbf{x} \neq \mathbf{0} \ , \quad r := |\mathbf{x}| \ .$$

(b) Prove that the function u defined in (3) is in $L^1_{\text{loc}}(\mathbb{R}^2)$.

Hint: As in Problem 4(a), it is enough to show that $\lim\limits_{\varepsilon\to 0^+} \int_{\{\varepsilon<|\mathbf{x}|<1\}} \big|\ln|\mathbf{x}|\big| \,\mathrm{d}\mathbf{x} < \infty$ and that $\lim\limits_{\varepsilon\to 0^+} \int_{\{1<|\mathbf{x}|<R\}} \ln|\mathbf{x}| \,\mathrm{d}\mathbf{x} < \infty$ for every $R > 1$; here $\mathrm{d}\mathbf{x} := \mathrm{d}x_1\,\mathrm{d}x_2$.

(c) Explain why u defined in (3) defines a distribution on $\mathscr{D}(\mathbb{R}^2)$.

(d) Prove the equality (2) in $\mathscr{D}'(\mathbb{R}^2)$.

Hint: Follow Example 7.7 on pages 380–382 of the book; use polar coordinates.

FFT ("Food for Thought") Problem 1.[1] [**Solutions of ODEs in** $\mathscr{D}'(\mathbb{R})$]
Show that the general solutions of the ODEs

$$xy_1' = 1 \ , \qquad x^2 y_2' = 0 \ , \qquad x^2 y_3' = 1 \ , \qquad y_4'' = \delta(x) \ , \qquad (x+1)y_5'' = 0$$

in $\mathscr{D}'(\mathbb{R})$ are the functions

$$y_1(x) = C_1 + C_2 H(x) + \ln|x| \ , \qquad y_2(x) = C_1 + C_2 H(x) + C_3\delta(x) \ ,$$

$$y_3(x) = C_1 + C_2 H(x) + C_3\delta(x) - \mathrm{P.v.}\frac{1}{x} \ ,$$

$$y_4(x) = C_1 + C_2 x + x\,H(x) \ , \qquad y_5(x) = C_1 + C_2 x + C_3 (x+1)\,H(x+1) \ .$$

FFT Problem 2. [**Problems with multiplication of functions and distributions**]
Consider the function $x \in C^\infty(\mathbb{R})$ and the distributions $\mathrm{P.v.}\frac{1}{x}, \delta \in \mathscr{D}'(\mathbb{R})$. Prove that $x\,\mathrm{P.v.}\frac{1}{x} = 1$, while $x\delta(x) = 0$, so $\left(x\,\mathrm{P.v.}\frac{1}{x}\right)\delta(x) = \delta(x)$, while $\big(x\delta(x)\big)\,\mathrm{P.v.}\frac{1}{x} = 0$. This implies that multiplication of functions and distributions cannot be commutative and associative.

[1] FFT, i.e., "Food for Thought", problems are for you to think about, but not to turn in with the regular homework.