MATH 5463 Homework 5 Due Thu, Mar 5

Problem 1. In this problem you will prove *Hilbert's inequalities*,

$$\int_0^\infty \int_0^\infty \frac{|f(x)|\,|g(y)|}{x+y}\,\mathrm{d}x\,\mathrm{d}y \le C_p \|f\|_p \|g\|_q \ , \tag{1}$$

where $f \in L^p([0,\infty))$ and $g \in L^q([0,\infty))$, where $p \in (1,\infty)$ and q are conjugate exponents, and C_p is the constant

$$C_p = \int_0^\infty \frac{\mathrm{d}\xi}{\xi^{1/p}(1+\xi)} \ . \tag{2}$$

Using contour integration, one can show that $C_p = \dfrac{\pi}{\sin(\pi/p)}$.

(a) Prove the so-called *Young's inequality*: let $\phi : [0,\infty) \to [0,\infty)$ be a continuous and strictly increasing function with $\phi(0) = 0$ and $\lim_{\xi\to\infty} \phi(\xi) = \infty$, and $\phi^{-1} : [0,\infty) \to [0,\infty)$ be its inverse. Then, for any $a \in [0,\infty)$ and $b \in [0,\infty)$,

$$ab \le \int_0^a \phi(\xi)\,\mathrm{d}\xi + \int_0^b \phi^{-1}(\eta)\,\mathrm{d}\eta \ .$$

One can also show (but you do not have to do this) that equality holds if and only if $b = \phi(a)$.

Hint: Interpret ab as the area of the rectangle in the (ξ,η)-plane with sides $[0,a]$ and $[0,b]$, $\int_0^a \phi(\xi)\,\mathrm{d}\xi$ as the area under the curve $\eta = \phi(\xi)$ for $\xi \in [0,a]$, and similarly for $\int_0^b \phi^{-1}(\eta)\,\mathrm{d}\eta$.

(b) Take $\phi(\xi) = \xi^{p-1}$ for $p \in (1,\infty)$ in Young's inequality, and derive the inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \ .$$

Compare this inequality with Lemma 6.1 from the book.

(c) If C_p is defined by (2), show that $C_p = C_q$. (Note also that this fact is consistent with the explicit expression for C_p given above.)

(d) Without loss of generality, you can assume that the functions f and g in (1) satisfy $\|f\|_p = 1$ and $\|g\|_q = 1$ (why?). First use the identity from (b) to obtain the inequality

$$\frac{|f(x)|\,|g(y)|}{x+y} \le \left(\frac{x\,|f(x)|^p}{p} + \frac{y\,|g(y)|^q}{q} \right) \frac{1}{(x+y)\,x^{1/p}\,y^{1/q}} \qquad \text{for } x \ge 0\ , \quad y \ge 0\ ,$$

then show that

$$\int_0^\infty \int_0^\infty \frac{x^{1-(1/p)}\,|f(x)|^p}{(x+y)\,y^{1/q}}\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty \int_0^\infty \frac{y^{1-(1/q)}\,|g(y)|^q}{(x+y)\,x^{1/p}}\,\mathrm{d}x\,\mathrm{d}y = C_p \ ,$$

and use this result to prove (1).

(e) Let $\mathbf{a} = (a_1, a_2, \ldots) \in \ell^p$, $\mathbf{b} = (b_1, b_2, \ldots) \in \ell^q$ (again, p and q are conjugate exponents, both in $(1, \infty)$). Use the same ideas as in (d) to prove the *Hilbert inequality for sums*,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|a_n|\,|b_m|}{m+n} \leq C_p \,\|\mathbf{a}\|_p \,\|\mathbf{b}\|_q \ .$$

The constant C_p is the same as above, but do not try to evaluate it as a sum – instead, use some easy bound on a sum by an integral.

Problem 2. Let $(X, \mathcal{M}, \mu)$ be a measure space, and f be a measurable function on it. Define the function $\Lambda_f : (0, \infty) \to [0, \infty]$ by

$$\Lambda_f(\alpha) = \mu\left(\{x \in X : |f(x)| > \alpha\}\right) \ .$$

(a) To warm up, find Λ_f if $X = \mathbb{R}$, μ is the Lebesgue measure, and $f(x) = \mathrm{e}^{-x}\,\chi_{[0,\infty)}(x)$.

In the rest of the problem, work for a general measure space $(X, \mathcal{M}, \mu)$!

(b) Show that Λ_f is decreasing and right continuous.

(c) Prove that if $|f| \leq |g|$, then $\Lambda_f \leq \Lambda_g$.

(d) If $|f_n|$ increases to $|f|$, then Λ_{f_n} increases to Λ_f.

(e) If $f = g + h$, then $\Lambda_f(\alpha) \leq \Lambda_g(\frac{\alpha}{2}) + \Lambda_h(\frac{\alpha}{2})$.

Food for thought. Think about the linear functionals acting on $C_c(X)$ if the measure μ is a Dirac measure concentrated at some point $x \in X$.