MATH 3423 Homework 6 Due Mon, 10/17/16

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function, and

$$\widehat{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-ikx}\, \mathrm{d}x$$

be its Fourier transform.

(a) Show that $\widehat{F}(-k) = \widehat{F}(k)^*$ for any $k \in \mathbb{R}$ (where the star stands for complex conjugation).

Hint: Note that, for any $x \in \mathbb{R}$, $f(x) \in \mathbb{R}$, therefore $f(x)^* = f(x)$. You can start like that:

$$\widehat{F}(k)^* = \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-ikx}\, \mathrm{d}x \right]^* = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)^* \left(\mathrm{e}^{-ikx}\right)^* \mathrm{d}x = \cdots \ .$$

(b) Based on your result from part (a), what can you say about $\widehat{F}(0)$?

Problem 2. Let α be a positive real number, and the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \mathrm{e}^{-\alpha|x|}\ ,$$

Use without proof that

$$\widehat{F}(k) = \sqrt{\frac{2}{\pi}}\, \frac{\alpha}{k^2 + \alpha^2}\ ,$$

as well as some property of Fourier transform (specify which property!) to show that the Fourier transform of the function

$$g(x) = x\mathrm{e}^{-\alpha|x|}$$

is

$$\widehat{G}(k) = -\sqrt{\frac{2}{\pi}}\, \frac{2\alpha i k}{(k^2 + \alpha^2)^2}\ .$$

Problem 3. Use the fact that the Fourier transform of the function $f(x) = \mathrm{e}^{-|x|}$ is

$$\widehat{F}(k) = \sqrt{\frac{2}{\pi}}\, \frac{1}{1 + k^2}$$

and Parseval's Theorem to show that

$$\int_0^{\infty} \frac{\mathrm{d}y}{(1 + y^2)^2} = \frac{\pi}{4}\ .$$

Problem 4. Consider the following initial value problem for the function $u(x, t)$:

$$\frac{1}{v}\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\ , \qquad x \in \mathbb{R}\ , \quad t > 0\ ,$$

$$u(x, 0) = f(x)\ ,$$

where v is a positive real constant.

(a) Let the Fourier transform with respect to x of $u(x,t)$ be

$$\widehat{U}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, \mathrm{e}^{-ikx}\, \mathrm{d}x$$

and its inverse be

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{U}(k,t)\, \mathrm{e}^{ikx}\, \mathrm{d}k \ .$$

Formulate an initial value problem for the function $\widehat{U}(k,t)$ of the form

$$\frac{\partial \widehat{U}}{\partial t}(k,t) = ?\ , \qquad k \in \mathbb{R}\ , \quad t > 0\ ,$$

$$\widehat{U}(k,0) = ?\ .$$

(b) Find the solution $\widehat{U}(k,t)$ of the initial value problem formulated in part (a).

(c) Use some of the properties of the Fourier transform (state explicitly which property you have used!) to show that the solution of the initial value problem is

$$u(x,t) = f(x+vt)\ .$$

Problem 5. The so-called *error function* is defined as

$$\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-\xi^2}\, \mathrm{d}\xi\ , \qquad z \in \mathbb{R}\ .$$

(a) Suppose that you want to write an initial-value problem for a first-order ordinary equations whose solution is $\mathrm{erf}(t)$; look for initial-value problem of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t,y)\ , \qquad t > 0$$

$$y(0) = y_0\ .$$

Find the explicit form of the function $f(t,y)$ and the numerical value of the constant y_0. Show your calculations and state explicitly what results you use in your derivation.

Hint: What is the derivative of $\mathrm{erf}(t)$? (Use the Fundamental Theorem of Calculus.)

(b) Is $\mathrm{erf}(t)$ an even or an odd function (or neither)? Prove your claim.

Hint: Note that the integrand, $\mathrm{e}^{-\xi^2}$, in the definition of $\mathrm{erf}(t)$ is an even function. Draw the graph of any even function and think how the value of the definite integral of this function from 0 to $-z$ is related to the value of the definite integral of this function from 0 to z; this will make the answer of my question totally obvious.

(c) If you know that $\int_{-\infty}^{\infty} \mathrm{e}^{-\xi^2}\, \mathrm{d}\xi = \sqrt{\pi}$, find the limits $\lim_{z\to-\infty} \mathrm{erf}(z)$ and $\lim_{z\to\infty} \mathrm{erf}(z)$.

(d) <u>In this part of the problem you only have to answer the question asked in the last sentence!</u>

In class we showed that the solution of the initial value problem for the heat equation on $\mathbb{R}$,

$$\begin{aligned} u_t(x,t) &= \alpha^2\, u_{xx}(x,t)\ , \qquad x \in \mathbb{R}\ , \quad t > 0\ , \\ u(x,0) &= g(x) \end{aligned} \tag{1}$$

(where $g(x)$ is a function satisfying $\lim\limits_{|x|\to\infty} f(x) = 0$), is given by

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{G}(k)\, \mathrm{e}^{-\alpha^2 k^2 t}\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k\ ,$$

with $\widehat{G} = \mathcal{F}\{g\}$. We came to this expression by showing that the Fourier transform of the function $u(x,t)$ (with respect to the spatial variable x) is

$$\widehat{U}(k,t) = \widehat{G}(k)\, \mathrm{e}^{-\alpha^2 k^2 t}\ . \tag{2}$$

Let us define the function

$$r(x,t) = \frac{1}{\sqrt{4\pi\alpha^2 t}}\, \mathrm{e}^{-x^2/(4\alpha^2 t)}\ . \tag{3}$$

The Fourier transform of $r(x,t)$ with respect to the spatial variable is

$$\widehat{R}(k,t) = \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-\alpha^2 k^2 t}\ .$$

This allows us to write the above expression for $\widehat{U}(k,t)$ as

$$\widehat{U}(k,t) = \sqrt{2\pi}\, \widehat{G}(k)\, \widehat{R}(k,t)\ .$$

Recall that the Convolution Theorem (equation (17) on page 851) says that

$$\mathcal{F}\{g * r\} = \sqrt{2\pi}\, \widehat{G}\, \widehat{R}\ .$$

Comparing this with (2), we see that the solution $u(x,t)$ of the initial value problem (1) is a convolution of the initial condition g and the function $r(x,t)$ given by (3) (again, the convolution is only over the spatial variable):

$$u = g * r\ ,$$

i.e.,

$$u(x,t) = \int_{-\infty}^{\infty} g(y)\, r(x-y,t)\, \mathrm{d}y = \frac{1}{\sqrt{4\pi\alpha^2 t}} \int_{-\infty}^{\infty} g(y)\, \mathrm{e}^{-(x-y)^2/(4\alpha^2 t)}\, \mathrm{d}y$$

(see equation (7) on page 857).

Write down the explicit form of the solution $u(x,t)$ for $g(x) = H(x)$, where H is the Heaviside function; express your expression for $u(x,t)$ in terms of the error function.

Information that will be provided in the exam

Definition of direct and inverse Fourier transform:

$$\hat{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-ikx}\, \mathrm{d}x \ , \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{F}(x)\, \mathrm{e}^{ikx}\, \mathrm{d}k \ .$$

Definition of convolution:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, \mathrm{d}y = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, \mathrm{d}y \ .$$

Properties of Fourier transform:

$$\mathrm{e}^{-iak}\hat{F}(k) = \mathcal{F}\{f(x-a)\} \ , \qquad \mathcal{F}\{\mathrm{e}^{iax}\, f(x)\} = \hat{F}(k-a) \ ;$$

$$\mathcal{F}\{f^{(n)}(x)\} = (ik)^n\, \hat{F}(k) \ , \qquad \mathcal{F}^{(n)}(k) = (-i)^n\, \mathcal{F}\{x^n\, f(x)\} \ ;$$

$$\mathcal{F}\{f * g\} = \sqrt{2\pi}\, \hat{F}(k)\, \hat{G}(k) \ .$$

Some useful Fourier transforms:

$f(x)$	$\hat{F}(k)$
1	$\sqrt{2\pi}\, \delta(k)$
e^{ik_0x}	$\sqrt{2\pi}\, \delta(k-k_0)$
$\mathrm{e}^{-a\lvert x\rvert}$ (for $a>0$)	$\sqrt{\frac{2}{\pi}}\, \frac{a}{k^2+a^2}$
$\mathrm{e}^{-a^2x^2}$ (for $a>0$)	$\frac{1}{\sqrt{2a^2}}\, \mathrm{e}^{-k^2/(4a^2)}$
$\frac{1}{x^2+a^2}$ (for $a>0$)	$\sqrt{\frac{\pi}{2a^2}}\, \mathrm{e}^{-a\lvert k\rvert}$
$\frac{1}{x}$	$-i\sqrt{\frac{2}{\pi}}\, \mathrm{sgn}(k)$
$H(x)$	$\sqrt{\frac{\pi}{2}}\, \delta(k) - \frac{i}{\sqrt{2\pi}k}$

Formula for differentiating an integral whose limits and integrand depend on the parameter α:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \int_{\phi(\alpha)}^{\psi(\alpha)} F(x,\alpha)\, \mathrm{d}x = F(\psi(\alpha),\alpha)\, \psi'(\alpha) - F(\phi(\alpha),\alpha)\, \phi'(\alpha) + \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial F}{\partial \alpha}(x,\alpha)\, \mathrm{d}x \ .$$