MATH 4433 **Homework 6** **Due 3/9/17 (Thursday)**

Abbott, Section 2.4:

Exercises 2.4.2, 2.4.3, 2.4.5, 2.4.6, 2.4.7 (pages 60, 61).

Remarks and hints:

- Exercise 2.4.2(b): justify.
- Exercise 2.4.3: justify.
- Exercise 2.4.5: to prove that $x_n^2 \geq 2$ for any $n \in \mathbb{N}$, you can use that

  $$x_{n+1}^2 = \frac{1}{4}\left(x_n^2 + 4 + \frac{4}{x_n^2}\right) = \frac{x_n^4 + 4}{4x_n^2} + 1 \ ,$$

  and $\dfrac{x_n^4 + 4}{4x_n^2} \geq 1$ because of the obvious inequality

  $$0 \leq (x_n^2 - 2)^2 = x_n^4 - 4x_n^2 + 4 \ .$$
- Exercise 2.4.6(b): show that both (x_n) and (y_n) are monotone; take the limit $n \to \infty$ in the defining relations.
- Exercise 2.4.7(a): explain why the sequence (y_n) is decreasing (the reason is very simple).

Abbott, Section 2.5:

Exercises 2.5.1(a,b,c), 2.5.2(b,c,d), 2.5.6, 2.5.9 (pages 65, 66).

Remarks and hints:

- Exercise 2.5.2: use the theorems from the text in your justification.
- Exercise 2.5.6: give a proof only for $b > 1$.
- Exercise 2.5.9: since $s = \sup S$, for every $\varepsilon > 0$ there exists a_n such that $s - \varepsilon < a_n$ (recall Lemma 1.3.8 on page 17 of Abbott); use this fact for $\epsilon = \frac{1}{k}$ with $k \in \mathbb{N}$ to construct the subsequence (a_{n_k}).

Additional Problem 1.

Suppose that $\lim a_n = a$, with $a > 0$. Directly from the definition of convergence, prove that there exists N such that $a_n > 0$ for every $n > N$.

Additional Problem 2.

Give an alternative proof of Theorem 2.3.3(iii) of Abbott that is based on the identity

$$a_n b_n - ab = (a_n - a)(b_n - b) + a(b_n - b) + b(a_n - a) \ .$$

Additional Problem 3.

Take for granted that the limit $\lim \left(1 + \frac{1}{n}\right)^n$ exists and is equal to e, to find the following limits:

(a) $a_n = \left(1 + \frac{1}{2n}\right)^{2n}$;

(b) $b_n = \left(1 + \frac{1}{n}\right)^{2n}$;

(c) $c_n = \left(1 + \frac{1}{n}\right)^{n-1}$;

(d) $d_n = \left(\frac{n}{n+1}\right)^{n}$;

(e) $e_n = \left(1 + \frac{1}{2n}\right)^{n}$;

(f) $f_n = \left(\frac{n+2}{n+1}\right)^{n+3}$.

Hint: In part (b), $b_n = \left(1 + \frac{1}{n}\right)^{2n} = \left[\left(1 + \frac{1}{n}\right)^{n}\right]^2$, which can be found by using the Algebraic Limit Theorem. Use similar ideas in all other parts; also, note that $\lim \left(1 + \frac{1}{n}\right) = 1$, which implies that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^k = 1$ for any $k \in \mathbb{N}$.