MATH 4443 Homework 6 Due 3/8/18 (Thursday)

Problem 1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let P be an arbitrary partition of the interval $[a, b]$.

(a) Use Lemma 7.2.4 to explain why $U(f) \geq L(f, P)$.

(b) Use your result in part (a) to show that $U(f) \geq L(f)$, which provides a proof of Lemma 7.2.6.

Problem 2. Consider the function $f : [0, 5] \to \mathbb{R} : x \mapsto x^2$. Take the uniform partition $P_n := \{0 = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = 5\}$ consisting of the points $x_k = \frac{5k}{n}$, $k = 0, 1, 2, \ldots, n$.

(a) Compute the value of $L(f, P_n)$. You may find useful that $\displaystyle\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

(b) Compute the value of $U(f, P_n)$.

(c) Show that, in this particular example, $\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0$.

Problem 3.

(a) Consider the piecewise linear functions f_n on $[0, 2]$ whose graph consists of segments of straight lines connecting the following pairs of points: $(0, 0)$ with $(\frac{1}{n}, n)$, $(\frac{1}{n}, n)$ with $(\frac{2}{n}, 0)$, and $(\frac{2}{n}, 0)$ with $(2, 0)$. In other words, $f_n : [0, 2] \to \mathbb{R}$ is given by

$$f_n(x) = \begin{cases} n^2 x \ , & x \in [0, \frac{1}{n}], \\ 2n - n^2 x \ , & x \in [\frac{1}{n}, \frac{2}{n}], \\ 0 \ , & x \in [\frac{2}{n}, 2]. \end{cases}$$

Since the area under the graph of f_n is simply an area of a triangle, you can find it by using elementary geometry, no need to integrate.

Find the pointwise limit, $\lim_{n\to\infty} f_n(x)$, and compare $\lim_{n\to\infty} \int_a^b f_n$ and $\int_a^b \lim_{n\to\infty} f_n$.

(b) Assume that, for each $n \in \mathbb{N}$, f_n is an integrable function on $[a, b]$. If (f_n) converges to f uniformly on $[a, b]$, prove that f is also integrable on $[a, b]$.

Hint: The Integrability Criterion from Theorem 7.2.8 will be useful. You can write

$$U(f, P) - L(f, P) = U(f, P) - L(f_n, P) + L(f_n, P) - L(f_n, P) + L(f_n, P) - L(f, P) \ ,$$

use the triangle inequality, the fact that the convergence of (f_n) is uniform, and the integrability of each f_n, to show that $\forall\, \varepsilon > 0$ given in advance, $U(f, P) - L(f, P) < \varepsilon$ for an appropriately chosen partition P.

Problem 4. A *tagged partition* $(P, \{x_k^*\})$ is a partition where in addition to a partition P, we choose a sampling point x_k^* in each subinterval $[x_{k-1}, x_k]$. The corresponding *Riemann sum* is defined as follows:

$$R(f, P) := \sum_{k=1}^{\infty} f(x_k^*)\, \Delta x_k \ , \qquad \Delta x_k = x_k - x_{k-1} \ .$$

Riemann originally defined the integral of f over $[a, b]$ as follows. A bounded function $f : [a, b] \to \mathbb{R}$ is *integrable* on $[a, b]$ with $\int_a^b f = A$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any tagged partition $(P, \{x_k^*\})$ with $\Delta x_k < \delta$ for all k, it follows that

$$|R(f, P) - A| < \varepsilon \ .$$

Show that if f satisfies the Riemann's definition for integrability, then f is integrable in the sense of Definition 7.2.7.

Problem 5. Let $f : [a, b] \to \mathbb{R}$ be increasing on the set $[a, b]$ (i.e., $f(x) \leq f(y)$ whenever $x < y$). Show that f is integrable on $[a, b]$.
Hint: Take a uniform partition (such that all Δx_k are the same), and you will obtain a very simple expression for $U(f, P) - L(f, P)$, which can easily be made smaller than any $\varepsilon > 0$.

Food for Thought: Abbott, Exercises 7.2.2, 7.2.3.