MATH 4093/5093 **Homework 6** **Due Mon, 11/01/10**

Problem 1. Consider the matrix

$$A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

Compute $\|A\|_1$, $\|A\|_2$, and $\|A\|_\infty$.
Hint: We derived the expression for $\|A\|_\infty$ in class, for $\|A\|_2$ use the Theorem on page 178 of the book, and the expression for $\|A\|_1$ is given in Exercise 7 on page 181 of the book (you do not need to derive it).

Problem 2. Consider the norms $\| \ \|_1$, $\| \ \|_2$, and $\| \ \|_\infty$ on $\mathbb{R}^n$.

(a) Prove that the norms $\| \ \|_1$ and $\| \ \|_\infty$ are equivalent.

(b) Prove that the norms $\| \ \|_2$ and $\| \ \|_\infty$ are equivalent.

(c) Prove that the norms $\| \ \|_1$ and $\| \ \|_2$ are equivalent.
Hint: Having solved parts (a) and (b), you can solve part (c) without any additional calculations.

Problem 3. Many theorems that hold in finite-dimensional spaces are not true in infinite-dimensional spaces. One can think of an infinite-dimensional space as a space of infinite sequences: $\mathbf{u} = (u_1, u_2, u_3, \ldots)$, where u_j are real numbers ($j \in \mathbb{N} = \{1, 2, 3, \ldots\}$). In this space we can define the ℓ_1, ℓ_2 and ℓ_∞ norms as follows:

$$\|\mathbf{u}\|_1 := \sum_{j\in\mathbb{N}} |u_j| \ , \qquad \|\mathbf{u}\|_2 := \left(\sum_{j\in\mathbb{N}} |u_j|^2\right)^{1/2} \ , \qquad \|\mathbf{u}\|_\infty := \sup_{j\in\mathbb{N}} |u_j|$$

($\inf_{j\in\mathbb{N}} a_n$ is the smallest number a such that $a_n \leq a$ for all $j \in \mathbb{N}$).

(a) Give an explicit example of a sequence $\mathbf{u}$ such that $\|\mathbf{u}\|_\infty < \infty$, but $\|\mathbf{u}\|_1$ is infinite.

(b) Give an explicit example of a sequence $\mathbf{u}$ such that $\|\mathbf{u}\|_\infty < \infty$, but $\|\mathbf{u}\|_2$ is infinite.

(c) Give an explicit example of a sequence $\mathbf{u}$ such that $\|\mathbf{u}\|_2 < \infty$, but $\|\mathbf{u}\|_1$ is infinite.
Hint: Think how you can use the following facts:

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} \ , \qquad \sum_{j=1}^{\infty} \frac{1}{j} = \infty \ .$$

Problem 4. Let A and B be nonsingular $n \times n$ matrices, α be a nonzero real number, and $\kappa(A)$ stand for the condition number of the matrix A.

(a) Show that $\kappa(AB) \leq \kappa(A)\kappa(B)$.

(b) Show that $\kappa(\alpha A) = \kappa(A)$.

Hint: These properties follow directly from the properties of norms.

Problem 5. Consider the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 2 \\ 1.001 & 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 3.001 \end{pmatrix}$$

The vector $\mathbf{x} = (1\ 1)^T$ is an exact solution, and the vector $\tilde{\mathbf{x}} = (3\ 0)^T$ is an approximate solution of the system.

(a) Compute the condition number $\kappa_\infty(A)$.

(b) Compute the error $\mathbf{e} = \tilde{\mathbf{x}} - \mathbf{x}$ and the residual $\mathbf{r} = A\tilde{\mathbf{x}} - \mathbf{b}$.

(c) Find the norms $\|\mathbf{e}\|_\infty$ and $\|\mathbf{r}\|_\infty$, the relative error $\|\mathbf{e}\|_\infty / \|\mathbf{x}\|_\infty$, and the relative residual $\|\mathbf{r}\|_\infty / \|\mathbf{b}\|_\infty$.

(d) Find the theoretical upper and lower bounds on $\|\mathbf{e}\|_\infty$ and the relative error $\|\mathbf{e}\|_\infty / \|\mathbf{x}\|_\infty$ from the Theorem on page 182 of the book, and show that they are satisfied by the actual values of the absolute and relative error computed in part (c). In all calculations use the ℓ_∞ norm.