MATH 5163 **Homework 6** **Due Fri, 2/28/14, 5 p.m.**

Problem 1. [Derivatives of distributions]

Find the first and the second derivatives of the following distributions from $\mathscr{D}'(\mathbb{R})$:

(a) $H(a-|x|)$ (where $a>0$);

(b) the "floor" function $\lfloor x \rfloor$, where $\lfloor x \rfloor$ is the smallest integer not exceeding x;

(c) the function

$$f(x) := \begin{cases} 0\ , & x \le 0\ , \\ \sin x\ , & x > 0\ . \end{cases}$$

Remark: No need to use test functions – feel free to use facts like $\frac{\mathrm{d}}{\mathrm{d}x}H(x-a) = \delta(x-a)$, etc.

Problem 2. [Convolution]

(a) For $\phi \in \mathscr{D}(\mathbb{R})$, find $\delta_a * \phi$.

(b) Show that $\delta_a * \delta_b = \delta_{a+b}$.

(c) Directly from the definition of convolution, prove that $(H*H)(x) = H(x)\,x$.

(d) If $u(x) = H(x)\,x^2$, find $H*u$.

Problem 3. [Convolution in $\mathscr{D}'(\mathbb{R}^2)$]

Let $F(x,t) = H(x)\,\delta(t)$ and $G(x,t) = \dfrac{H(t)}{2\sqrt{\pi t}}\,\mathrm{e}^{-x^2/(4t)}$. Show that

$$(F*G)(x,t) = \frac{H(t)}{\sqrt{2\pi}} \int_{-\infty}^{x/(2\sqrt{t})} \mathrm{e}^{-z^2/2}\,\mathrm{d}z\ .$$

Problem 4. [Fourier transform of sign and H]

(a) Prove that $\mathcal{F}(\mathrm{sign})(\xi) = \dfrac{2}{\mathrm{i}}\,\mathrm{P.v.}\dfrac{1}{\xi}$, where the "sign" function is defined as

$$\mathrm{sign}(x) := \begin{cases} -1\ , & x<0\ , \\ 0\ , & x=0\ , \\ 1\ , & x>0\ . \end{cases}$$

Hint: Note that $\mathrm{sign}' = 2\delta$, and transform this to obtain $\xi\,\widehat{\mathrm{sign}}(\xi) = -2\mathrm{i}$. Use Example 7.12 on page 386 of the book and recall that $\widehat{\mathrm{sign}}$ is odd while δ is even.

(b) Show that $\mathcal{F}(H)(\xi) = \pi\delta(\xi) + \frac{1}{\mathrm{i}}\,\mathrm{P.v.}\frac{1}{\xi}$.

Hint: The easiest way to prove this is to express H in terms of sign and to use (a).

Problem 5. [Computing integrals by Parseval's identity]

(a) Show that $\mathcal{F}[H(a-|x|)] = 2\,\frac{\sin a\xi}{\xi}$ (where $x \in \mathbb{R}$ and $a = \text{const} > 0$).

(b) Use Parseval's identity and the result from part (a) to show that

$$\int_0^\infty \frac{\sin ax\,\sin bx}{x^2}\,\mathrm{d}x = \frac{\pi}{2}\,\min(a,b)\ .$$

Problem 6. [Heisenberg uncertainty principle on $\mathbb{R}$]
Let $f \in \mathscr{S}(\mathbb{R})$. From the general theory we know that $f \in L^2(\mathbb{R})$ and, therefore, its Fourier transform $\hat{f}$ is also in $L^2(\mathbb{R})$.

(a) Show that $\|f\|^2_{L^2(\mathbb{R})} := \int_{\mathbb{R}} |f(x)|^2\,\mathrm{d}x = -2\int_{\mathbb{R}} x\,f(x)\,f'(x)\,\mathrm{d}x$.

Hint: Consider the identity $|f(x)| = |f(x)|^2\,\frac{\mathrm{d}}{\mathrm{d}x}x$ together with integration by parts.

(b) Use part (a) to prove that $\|f\|^2_{L^2(\mathbb{R})} \le 2\,\|xf\|_{L^2(\mathbb{R})}\,\|f'\|_{L^2(\mathbb{R})}$.

(c) Use part (b) and some basic properties of the Fourier transform to conclude that

$$\|f\|^2_{L^2(\mathbb{R})} \le 4\pi\,\|xf\|_{L^2(\mathbb{R})}\,\|\xi\hat{f}\|_{L^2(\mathbb{R})}\ .$$

This result can be interpreted that if, say, $\|f\|_{L^2(\mathbb{R})} = 1$, then the function f and its Fourier transform $\hat{f}$ cannot be simultaneously too "localized" around 0. In the language of quantum physics, this means that it is impossible to measure simultaneously the position and the momentum of a particle with arbitrary accuracy.