MATH 5763 Homework 6 Due Fri, 3/11/11

Problem 1. Recall that in class we showed that a Poisson process $\{N_t\}_{t\geq 0}$ of rate λ can be constructed as follows. Let $X_1, X_2, \ldots$ be i.i.d. exponential random variables with parameter λ, i.e., $X_m \sim \text{Exp}(\lambda)$; these random variables play the role of the inter-event time intervals (i.e., the time intervals between two consecutive events). Then the time of the nth event is $T_n = \sum_{m=1}^{n} X_m$, and the Poisson process $\{N_t\}$ can be obtained by $N_t = \max\{n \in \mathbb{Z}_+ : T_n \leq t\}$. We proved that the time T_n of the nth event is a $\Gamma(n,\lambda)$ random variable. It is not too difficult to prove by induction that the c.d.f. of $T_n \sim \Gamma(n,\lambda)$ is

$$F_{\Gamma(n,\lambda)}(t) = \begin{cases} 0\ , & \text{if } t < 0\ , \\ 1 - \mathrm{e}^{-\lambda t} \displaystyle\sum_{m=0}^{n-1} \frac{(\lambda t)^m}{m!}\ , & \text{if } t \geq 0\ . \end{cases}$$

In this problem you will relate the Poisson process $\{N_t\}$ with the Γ random variables in a way different from the one we used in class.

(a) Explain why $N_t = n$ is equivalent to $T_n \leq t < T_{n+1}$.

(b) For any value of $t \geq 0$, how are the events $\{T_{n+1} \leq t\}$ and $\{T_n \leq t\}$ related?

(c) For any value of $t \geq 0$, express the event $\{N_t = n\}$ in terms of the events $\{T_{n+1} \leq t\}$ and $\{T_n \leq t\}$.

(d) Assume that you know that $T_n \sim \Gamma(n,\lambda)$ and also know the explicit expression for $F_{\Gamma(n,\lambda)}$ (given above), and use your result from part (c) to find $\mathbb{P}(N_t = n)$.

Problem 2. Let W_t and M_t be the number of women, respectively men, entering a big store in the time interval $[0,t]$. Assume that $W = \{W_t\}_{t\geq 0}$ and $M = \{M_t\}_{t\geq 0}$ are independent Poisson processes with intensities ω and μ, respectively.

(a) Prove that the total number of people, $N_t := W_t + M_t$, entering the store forms a Poisson process and find its intensity. Explain briefly why your result is obvious.

(b) Find the conditional probability $\mathbb{P}(M_t = m \,|\, N_t = n)$, for $0 \leq m \leq n$, and show that $M_t \sim \text{Bin}(N_t, \frac{\mu}{\omega+\mu})$. Why is this result obvious? What is the conditional expectation $\mathbb{E}[M_t \,|\, N_t]$? (To answer the last question, you can use what you know about binomial random variables.)

Hint: To find $\mathbb{P}(M_t = m \,|\, N_t = n)$, use that, for a Poisson process with rate λ, $p_{ij}(t) = \mathrm{e}^{-\lambda t}\frac{(\lambda t)^{j-i}}{(j-i)!}$ for $j \geq i$, and $p_{ij}(t) = 0$ for $j < i$ (which follows easily from Problem 1(d) above). Also, recall that the conditional expectation $\mathbb{E}[M_t \,|\, N_t]$ will be a function of N_t and possibly some parameters related to the problem.

Problem 3. Let $N = \{N_t : t \geq 0\}$ be a (time-homogeneous) Poisson process with rate λ. Define the *flip-flop process* $X = \{X_t : t \geq 0\}$ with state space $\mathscr{X} = \{0, 1\}$ by

$$X_t = \frac{1}{2} + (-1)^{N_t} \left[X_0 - \frac{1}{2} \right] ,$$

where X_0 is a random variable with values in $\mathscr{X}$ independent of the Poisson process N. In other words, the flip-flop process switches between the states 0 and 1 at each event of N. Since N is a Markov chain, X is also a Markov chain. Let $\mathbf{P}_t = \begin{pmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{pmatrix}$ be the stochastic semigroup of the process X, and $\mathbf{G}$ be the generator of $\mathbf{P}_t$.

(a) Find the short-time transition probabilities $p_{ij}(h) = \mathbb{P}(X_{t+h} = j | X_t = i)$ and show that the generator of the stochastic process X is $\mathbf{G} = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}$.

(b) To find the time evolution of the chain X – i.e., to find the stochastic semigroup

$$\mathbf{P}_t = \mathrm{e}^{t\mathbf{G}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbf{G}^n ,$$

one needs to find $\mathbf{G}^n$ for $n \in \mathbb{N}$ (of course, $\mathbf{G}^0 = \mathbf{I}$, the identity matrix).

One way to compute $\mathbf{G}^n$ is to diagonalize it by a similarity transformation, $\widetilde{\mathbf{G}} = \mathbf{M}^{-1}\mathbf{G}\mathbf{M}$, using the tricks learned in Problem 4 of Homework 3. Then compute the nth power of the diagonal matrix $\widetilde{\mathbf{G}}$ (which is very easy), and finally use that $\widetilde{\mathbf{G}}^n = \mathbf{M}^{-1}\mathbf{G}^n\mathbf{M}$, so that $\mathbf{G}^n = \mathbf{M}\widetilde{\mathbf{G}}^n\mathbf{M}^{-1}$. In fact, one can directly compute the diagonal matrix $\mathrm{e}^{t\widetilde{\mathbf{G}}}$, and then to use that

$$\mathbf{P}_t = \mathrm{e}^{t\mathbf{G}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbf{G}^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbf{M}\widetilde{\mathbf{G}}^n\mathbf{M}^{-1} = \mathbf{M} \sum_{n=0}^{\infty} \frac{t^n}{n!} \widetilde{\mathbf{G}}^n\mathbf{M}^{-1} = \mathbf{M}\, \mathrm{e}^{t\widetilde{\mathbf{G}}}\mathbf{M}^{-1} .$$

At the end, you should obtain that $\mathbf{P}_t = \begin{pmatrix} \frac{1}{2}(1 + \mathrm{e}^{-2\lambda t}) & \frac{1}{2}(1 - \mathrm{e}^{-2\lambda t}) \\ \frac{1}{2}(1 - \mathrm{e}^{-2\lambda t}) & \frac{1}{2}(1 + \mathrm{e}^{-2\lambda t}) \end{pmatrix}$, but I would like to see the details of your computations.

(c) Now you will find $\mathbf{P}_t$ directly, without using the generator $\mathbf{G}$. (Of course, you have to pretend that you don't know the answer.) One can do this using several methods.

The standard method is to solve the forward Kolmogorov equations, $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}_t = \mathbf{P}_t\mathbf{G}$, with appropriate initial conditions; we will do this in class, so you don't need to do it here.

A trickier method for computing $\mathbf{P}_t$ (which works in this particular problem) is the following. Note that $p_{01}(t) = \mathbb{P}(X_{s+t} = 1 \,|\, X_s = 0)$ is equal to the probability that there were an odd number of events of the Poisson process N of intensity λ in the interval $(s, s+t]$. Using the explicit expression for the probability of exactly k events of a Poisson process to occur in a time interval of length t, compute $p_{01}(t)$. Having computed $p_{01}(t)$, you can easily find $p_{00}(t)$, $p_{10}(t)$, and $p_{11}(t)$ using the same tricks as in part (c). (If you have found $p_{00}(t)$, $p_{10}(t)$, and $p_{11}(t)$ in part (c), there is no need to do it again here.)

Hint: Note that $\displaystyle\sum_{j\text{ odd}} \frac{\alpha^j}{j!} = \sum_{k=0}^{\infty} \frac{\alpha^{2k+1}}{(2k+1)!} = \frac{1}{2}\left(\mathrm{e}^{\alpha} - \mathrm{e}^{-\alpha}\right).$

(d) Find the stationary distribution $\boldsymbol{\pi}$ by using the generator $\mathbf{G}$.

(e) Now assume that initially the chain X is in state 0 (i.e., that $X_0 = 0$). Determine the probability distribution $\mathbf{p}(t) = (p_0(t)\;\; p_1(t))$ (where $p_j(t) = \mathbb{P}(X_t = j)$) of the chain X at time t by using your results above. As t goes to infinity, does $\mathbf{p}(t)$ tend to the stationary distribution $\boldsymbol{\pi}$?

(f) Define the generating functions $\Delta_i(\xi, t) := \displaystyle\sum_{j=0}^{1} p_{ij}(t)\,\xi^j$, and show that Δ_i satisfies the first-order partial differential equation $\frac{\partial \Delta_i}{\partial t} + 2\lambda(\xi - 1)\frac{\partial \Delta_i}{\partial \xi} = \lambda(\xi - 1)$. Since the proofs for Δ_0 and Δ_1 are essentially the same, give a proof only for Δ_0. You have to do this by using the forward Kolmogorov equations, $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}_t = \mathbf{P}_t\mathbf{G}$, i.e.,

$$\begin{pmatrix} p'_{00}(t) & p'_{01}(t) \\ p'_{10}(t) & p'_{11}(t) \end{pmatrix} = \begin{pmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{pmatrix} \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix} .$$

(g) What are the initial conditions that Δ_0 and Δ_1 must satisfy? Why? (Recall that we assumed that initially the chain is in state 0.)

One can show that the solution of the PDE above and the corresponding initial conditions is

$$\Delta_0(\xi, t) = \frac{1}{2}\left[1 + \xi + (1 - \xi)\mathrm{e}^{-2\lambda t}\right] , \qquad \Delta_1(\xi, t) = \frac{1}{2}\left[1 + \xi - (1 - \xi)\mathrm{e}^{-2\lambda t}\right] .$$

(h) From the very definition of $\Delta_i(\xi, t)$, show that

$$\mathbb{E}[X_t | X_0 = i] = \frac{\partial \Delta_i}{\partial \xi}(1, t)$$

and

$$\mathrm{Var}[X_t | X_0 = i] = \mathbb{E}[X_t^2 | X_0 = i] - \mathbb{E}[X_t | X_0 = i]^2 = \frac{\partial^2 \Delta_i}{\partial^2 \xi}(1, t) + \frac{\partial \Delta_i}{\partial \xi}(1, t) - \left[\frac{\partial \Delta_i}{\partial \xi}(1, t)\right]^2 .$$

Hint: Recall Problem 4 from Homework 5.

(i) Use the concrete expressions for the generating functions $\Delta_i(\xi, t)$ of the flip-flop problem (written in part (g)) in order to find the conditional expectation $\mathbb{E}[X_t \,|\, X_0 = 0]$ and the conditional variance $\mathrm{var}[X_t \,|\, X_0 = 0]$, and sketch $\mathbb{E}[X_t \,|\, X_0 = 0]$ and $\mathrm{var}[X_t \,|\, X_0 = 0]$ as functions of t. Do your results look reasonable in the limiting cases $t \to 0+$ and $t \to \infty$? Explain briefly.