Homework 7 – Additional problems, part 2

Problem 2. In this problem you will study numerical approximations to the value of the integral

$$I := \int_0^2 (1+x^3)^{\frac{1}{3}}\,\mathrm{d}x\ ,$$

whose exact value is

$$I = 2\ {}_2F_1\left(-\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{4}{3}, -8\right) = 2.71832368232376\ldots\ .$$

Here ${}_2F_1\,(a,b,c,z)$ is the so-called *Gaussian hypergeometric function* – see, e.g.,

`https://en.wikipedia.org/wiki/Hypergeometric_function`

(a) Compute the value of the value of T_5 (i.e., the approximation of the integral I given by the Trapezoidal Rule with $n = 5$.

(b) Compute the value of the rigorous upper bound on the error $|T_5 - I|$ given in the Error Bounds box on page 534. You may use the following facts:

$$\frac{\mathrm{d}}{\mathrm{d}x}\,(1+x^3)^{\frac{1}{3}} = \frac{x^2}{(1+x^3)^{\frac{2}{3}}}\ , \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\,(1+x^3)^{\frac{1}{3}} = \frac{2x}{(1+x^3)^{\frac{5}{3}}}\ , \qquad \frac{\mathrm{d}^3}{\mathrm{d}x^3}\,(1+x^3)^{\frac{1}{3}} = \frac{2-8x^3}{(1+x^3)^{\frac{8}{3}}}\ .$$

Please write your computations and reasoning in detail.

(c) Compute the true value of the error $|T_5 - I|$ by using the value of T_5 found in part (a). Compare with the bound computed in part (b). Comment briefly.

(d) Find the minimum value of $n \in \mathbb{N}$ that is needed to achieve accuracy $|T_5 - I| \leq 10^{-16}$.