MATH 3413 **Homework 7** **Due Thu, Mar 29, 2012**

Sec. 7.4: problems 3, 22, 23, 29, 36.

Hint to Problem 7.4/29: To find the Laplace transform of the product $tx'(t)$, use Theorem 2 on page 476 and the Corollary on page 454 to obtain the following:

$$\mathcal{L}\{tx'(t)\}(s) = -\mathcal{L}\{-tx'(t)\}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{x'\}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}[sX(s) - x(0)] \ ,$$

and similarly for $tx''(t)$; see also Example 5 on page 477.

Sec. 7.5: problems 2, 5, 25.

Sec. 7.6: problems 5, 14.

Additional problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is differentiable infinitely many times. Recall that the nth derivative, $\delta_a^{(n)}(t) := \frac{\mathrm{d}^n}{\mathrm{d}t^n}\delta_a(t)$, of $\delta_a(t)$ is defined by the formula

$$\int_{\mathbb{R}} \delta_a^{(n)}(t)\, f(t)\, \mathrm{d}t := (-1)^n f^{(n)}(a) \ . \qquad (1)$$

The motivation for this definition came from treating the derivatives of $\delta_a(t)$ as ordinary functions, integrating by parts, and using that at $\pm\infty$ the function $\delta_a(t)$ is "equal" to zero. In this problem you will give a meaning of the formal definition of a derivative of $\delta_a(t)$ that looks like the derivative of an "ordinary" function:

$$\frac{\widetilde{\mathrm{d}}}{\mathrm{d}t}\,\delta_a(t) \quad \text{“}:=\text{''} \quad \lim_{h\to 0} \frac{\delta_a(t+h) - \delta_a(t)}{h} \ ; \qquad (2)$$

here the tilde over the derivative sign simply means that this definition is different from the definition (1) of the derivative of $\delta_a(t)$. Inspired by (2), define

$$\int_{\mathbb{R}} \left(\frac{\widetilde{\mathrm{d}}}{\mathrm{d}t}\,\delta_a(t) \right) f(t)\, \mathrm{d}t := \lim_{h\to 0} \int_{\mathbb{R}} \frac{\delta_a(t+h) - \delta_a(t)}{h} f(t)\, \mathrm{d}t \ . \qquad (3)$$

(a) Change the variable t in $\int_{\mathbb{R}} \delta_a(t+h)\, f(t)\, \mathrm{d}t$ to $z = t+h$ to compute this integral.

(b) Using your result from part (a), find $\displaystyle\int_{\mathbb{R}} \frac{\delta_a(t+h) - \delta_a(t)}{h} f(t)\, \mathrm{d}t$.

(c) Find $\displaystyle\int_{\mathbb{R}} \left(\frac{\widetilde{\mathrm{d}}}{\mathrm{d}t}\,\delta_a(t) \right) f(t)\, \mathrm{d}t$ defined by (3), and compare your result with $\int_{\mathbb{R}} \delta_a'(t)\, f(t)\, \mathrm{d}t$ given by equation (1). Discuss briefly your findings.